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AVIATION ENGINE PERFORMANCE.*

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In what follows it is proposed to discuss what performance may reasonably be expected of aviation engines. Altitude considered merely as a measure of "distance up" obviously is not an important factor, but the engine is very much concerned with the pressure and temperature of the air at any "distance up." Such information is given in Fig. 1. Any relation between pressure, temperature, and altitude must be considered only as approximate, as at any altitude pressure and temperature vary from day to day and from hour to hour. There is therefore an increasing and very desirable tendency to plot performance not against altitude, but against the density (weight per unit volume) or pressure of the atmosphere. In either case, for reasons which will be made clear later, the temperature should be plotted or recorded.

The conventional aviation engine is of the internal combustion type, its source of power being the combustible mixture of fuel and air with which it is supplied. The work which it can do per cycle depends upon the quality (fuel-air ratio) and quantity of mixture which it receives. An engine gathers its charge by the "cylinder full," that is, by volume rather than by weight. Inasmuch as in the ordinary explosive mixture of gasoline and air the

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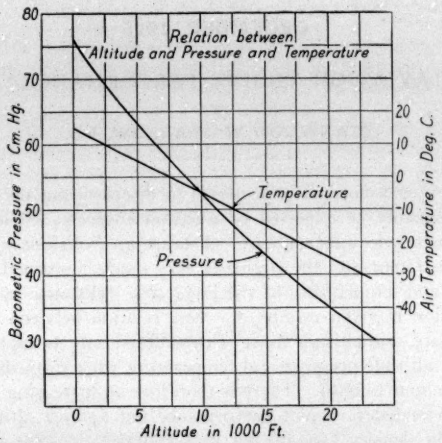
vapor volume constitutes only from 1 to 2.5 per cent. of the total volume of the charge, it is evident that the quality of the mixture can be varied over a wide range without affecting to any great extent the amount of air entering the engine. Hence the work which the engine can do per cycle depends ultimately upon the amount of air which it receives. Fig. 2 shows pictorially the relation between volumes of fuel vapor and air and draws attention to

FIG. 1.



the fact that a large proportion of the air is inert gas, the oxygen alone supporting combustion.

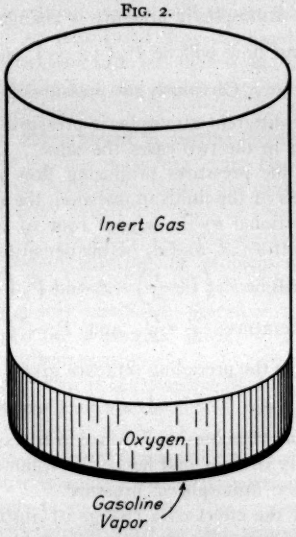
The term "volumetric efficiency," as used in this paper, may be defined as the ratio between the volume of charge received per cycle measured at the temperature and pressure existing at the entrance to the carburetor and the piston displacement. Both volumetric efficiency and air density (weight per unit volume) affect the weight of charge received by the engine per cycle. The former is governed very largely by engine design, the latter not at all.

The influences of pressure and temperature upon density are known. The weight of a unit volume of air is directly proportional to its absolute pressure and inversely proportional to its

absolute temperature. The influence of pressure and temperature upon volumetric efficiency, therefore, must be investigated.

The many tests which have been made in the altitude laboratory of the Bureau of Standards show no evidence that barometric pressure has an influence upon volumetric efficiency. No such influence would be anticipated on theoretical grounds. One example will serve to show the factors involved. Consider that the piston has completed a small portion of its suction stroke and that the intake valve is now opened. The velocity of flow through the valve will depend upon the magnitude of the pressure difference. Assume that the pressure upon the outside of the valve is atmospheric, P_1, and that the pressure within the cylinder is P_2. As P_2 is supposedly the result of the adiabatic expansion of the gases in the clearance space that were presumably at atmospheric pressure at the beginning of the intake stroke, P_2 is then proportional to this atmospheric pressure P_1, or say $P_2 = \frac{P_1}{m}$. The difference

FIG. 2.



Relative volume of constituents of a combustible mixture of fuel and air.

producing flow is $P_1 - P_2 = P_1\left(1 - \frac{1}{m}\right)$. If we assume the intake valve again closed while the pressure difference is still $P_1 - P_2$, the piston again given a small displacement and the valve again opened, the pressure will still be proportional to P_1, but with another factor, say $\frac{1}{n}$. Now if the initial displacement is kept the same but the atmospheric pressure is changed to say P_3, the pressure producing flow will be $P_3\left(1 - \frac{1}{m}\right)$ and for the second step $P_3\left(1 - \frac{1}{n}\right)$, etc. Obviously the pressures producing flow are not the same for different atmospheric pressures, but neither are the fluids flowing in the two cases the same.

If, however, the pressures producing flow are expressed in terms of the heads of the fluids in question, the relation that flow velocity is proportional to the square root of the head may be applied. If, therefore, d_1 and d_3 be the densities in the instances cited, the heads will be $P_1\left(1 - \frac{1}{m}\right) \div d_1$ and $P_3\left(1 - \frac{1}{m}\right) \div d_3$. At a constant temperature $\frac{d_1}{d_3} = \frac{P_1}{P_3}$ and $P_3 = \frac{P_1 d_3}{d_1}$. Substituting this value for P_3 in the preceding sentence gives $\frac{P_1 d_3}{d_1}\left(1 - \frac{1}{m}\right) \div d_3 = P_1\left(1 - \frac{1}{m}\right) \div d_1$. The heads are thus seen to be equal in the two cases and equal velocities of flow are to be expected. In other words, the velocity of flow and hence the volumetric efficiency is independent of the atmospheric pressure.

Now consider the effect of a change of absolute temperature from T_1 to T_3, the pressures remaining constant. If the densities be d_1 and d_3, the heads will be $\frac{P_1}{d_1}$ and $\frac{P_1}{d_3}$. At a constant pressure $\frac{d_1}{d_3} = \frac{T_3}{T_1}$. Hence $\frac{P_1}{d_3} \div \frac{P_1}{d_1} = \frac{d_1}{d_3} = \frac{T_3}{T_1}$. It is seen that the heads vary directly as the absolute temperature, and as the velocity of flow is proportional to the square root of the head, it follows that with the same pressure difference the velocity of flow will vary directly as the square root of the absolute temperature. The "joker" in the preceding statement is the phrase "the same pressure difference." If the flow into the engine during a given portion of the suction stroke be increased, there will be an increase in the pressure within the cylinder and hence the pressure differ-

ence during the remainder of the stroke will be decreased. The limiting case occurs when the engine speed is so slow and the time available for flow into the cylinder so long that the volumetric efficiency is nearly 100 per cent. at any entrance air temperature. Under such conditions the average air velocity can be increased only a negligible amount by an increase in entering air temperature. From the foregoing, one would expect that under all conditions the change in velocity of flow produced by a change in temperature would be somewhat less than would be predicted from the square root relation and that under certain circumstances the change would be negligible.

In the engine the problem is further complicated by other factors, such as the heating or cooling of the charge in its passage to the cylinder or in the cylinder before the closing of the intake valve. Over 1600 tests, covering a range of entrance air temperatures from $-30°$ C. to $+60°$ C., indicate that the combined effect of all the factors entering the problem is to give results remarkably close to those which would be calculated by the square root relation. It seems justifiable, therefore, to state that the velocity of flow and the volumetric efficiency vary as the square root of the absolute temperature.

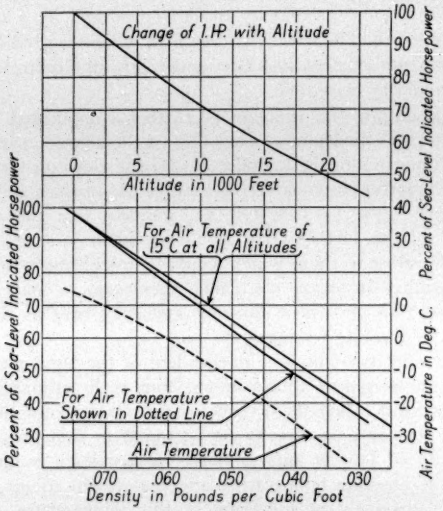
If the velocity of flow is independent of the atmospheric pressure and the weight of air per unit volume is directly proportional to this pressure, then the weight of air received by the engine per cycle will also be directly proportional to this pressure. If the velocity of flow is approximately proportional to the square root of the absolute temperature and the weight of air per unit volume is inversely proportional to this temperature, then the weight of air received by the engine per cycle will be approximately inversely proportional to the square root of the absolute temperature.

Whether or not indicated horsepower, the power developed in the engine cylinder, will vary directly as the weight of air received per cycle, depends upon whether or not the completeness with which the air is utilized is affected by changes of pressure or temperature. Tests have shown no such effect of any considerable magnitude. Hence indicated horsepower can be expected to vary directly as the barometric pressure and inversely approximately as the square root of the absolute temperature.

If the foregoing conclusions are correct, and if pressure and

temperature vary with altitude as shown in Fig. 1, then horsepower will vary with altitude as shown in Fig. 3. The lower portion of the figure shows the same data plotted against density, and also the variation that would result if the temperature remained constant at its sea-level value. These curves, showing as they do a difference in power at the same density, emphasize

FIG. 3.



the need for giving complete temperature data when plotting results *versus* pressure or density.

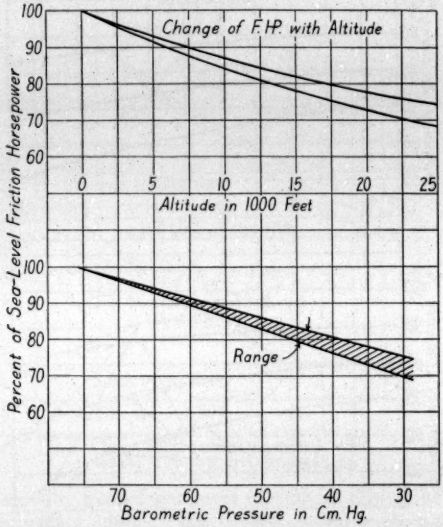
From a practical standpoint, one is not so much concerned with how much power is developed in the cylinder as in how much power is delivered to the propeller. To predict this, the brake horsepower, something must be known about friction, for brake horsepower is the difference between indicated horsepower and friction horsepower. For the purposes of this paper it is particularly important to know how friction horsepower is affected by atmospheric pressure and temperature.

The definition of friction horsepower will be evident from the definition of brake horsepower just given. As thus defined, it

includes the power expended in drawing in and exhausting the charge, and in driving the auxiliaries, such as pump, magneto, etc., as well as the power expended in overcoming what can strictly be termed engine friction. It seems probable that the change of friction horsepower with change of atmospheric pressure is very largely the result of a change in pumping loss.

There is no evidence that a change in air temperature has a

FIG. 4.

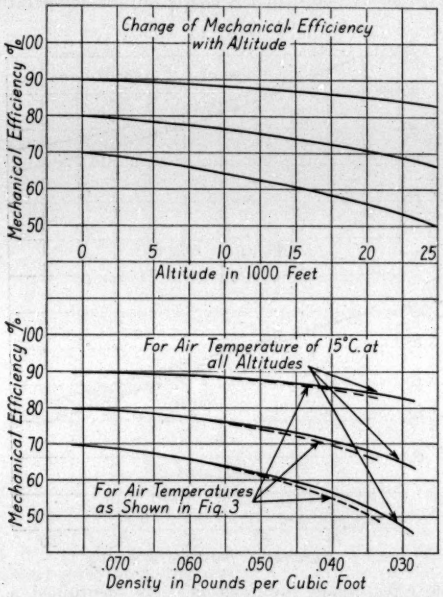


measurable effect upon the friction. As mentioned a moment ago, pressure does have an effect and the magnitude of its influence is indicated in Fig. 4. The figure is somewhat unsatisfactory, for while the changes in the friction horsepower of the many engines tested have been within the limits shown, there is no positive assurance that the friction of other engines may not be affected to a greater or less extent by barometric pressure.

At one-half of atmospheric pressure the decrease in friction is shown by Fig. 4 to be of the order of 25 per cent., whereas the indicated horsepower has decreased nearly 50 per cent. This

shows why there is a decrease in mechanical efficiency with increase in altitude. Fig. 5 shows mechanical efficiencies plotted in accordance with a series of assumptions, the first being that indicated horsepower varies as shown in previous figures, the second being that friction varies in accordance with the lower

FIG. 5.

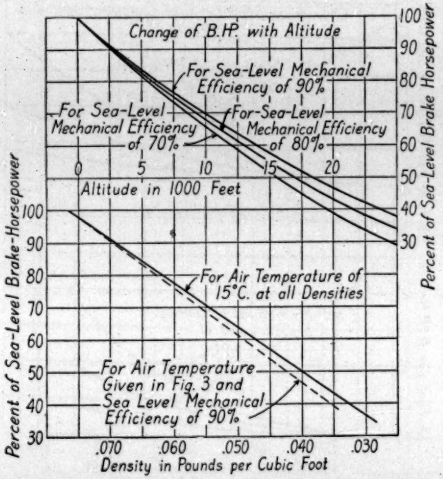


line of Fig. 4 and the third being that at sea-level the magnitude of the friction is such that the mechanical efficiency is either 90 per cent. or 80 per cent. or 70 per cent. It may be well to point out in connection with this figure that the friction at a given air density is dependent upon the air temperature. This is not inconsistent with the previously made statement that a change in air temperature has no measurable effect upon friction. If atmospheric conditions are the same as regards density but different as regards

barometric pressure. The difference in friction under such conditions is a consequence of the difference in pressure.

Upon the assumptions just mentioned Fig. 6 is also based. With a sea-level mechanical efficiency of 90 per cent., 47 per cent. of sea-level brake horsepower is obtained at 20,000 feet and 30 per cent. at 30,000 feet. There is some justification, therefore,

FIG. 6.

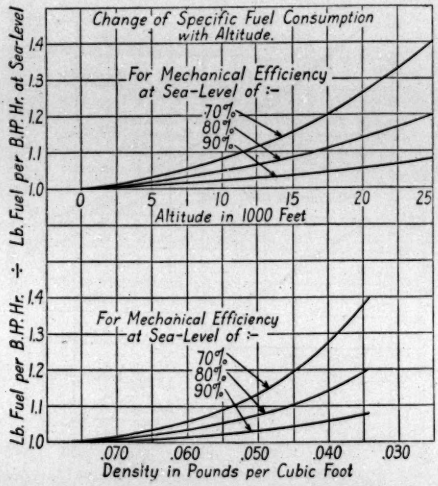

Change of B.H.P. with Altitude
For Sea-Level Mechanical Efficiency of 90%
For Sea-Level Mechanical Efficiency of 70%
For Sea-Level Mechanical Efficiency of 80%
0 5 10 15 20
Altitude in 1000 Feet
For Air Temperature of 15°C. at all Densities
For Air Temperature Given in Fig. 3 and Sea Level Mechanical Efficiency of 90%
.070 .060 .050 .040 .030
Density in Pounds per Cubic Foot
Percent of Sea-Level Brake-Horsepower
Percent of Sea-Level Brake Horsepower
100 90 80 70 60 50 40 30

for the frequently quoted statement that at an altitude of 20,000 feet approximately one-half sea-level power is developed and at 30,000 feet approximately one-third.

Indicated thermal efficiency and hence specific fuel consumption in pounds per indicated horsepower hour are shown by experiment and would be expected from theory to be independent of air pressure and temperature. The former, multiplied by the mechanical efficiency, gives brake thermal efficiency and the latter divided by mechanical efficiency gives specific fuel consumption in pounds per brake horsepower hour. Both, being dependent upon mechanical efficiency, vary with altitude, and Fig. 7 shows the extent of this variation.

The statement that indicated thermal efficiency and specific fuel consumption do not vary with altitude presupposes that the fuel-air ratio required for maximum power or any percentage of maximum power is the same for each altitude. Theory indicates and experiments demonstrate the validity of this supposition. With a fuel less volatile than aviation gasoline, changes in alti-

FIG. 7.



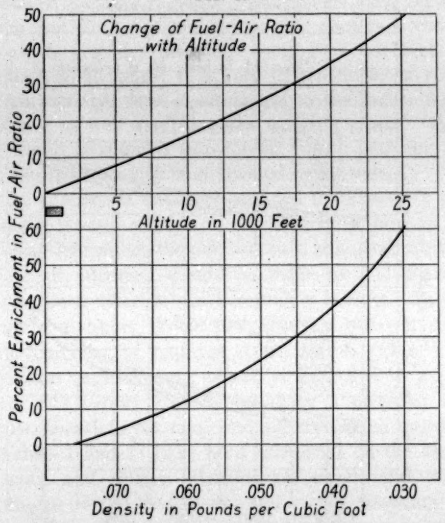
tude might bring about changes in the equality of distribution which would make a change in fuel-air ratio desirable. At present, however, constancy is desired and its attainment is the task that confronts the carburetor designer.

As has been implied, the carburetor delivers a constant fuel-air ratio only under compulsion. Its normal tendency is to deliver a mixture whose richness increases with the altitude. The amount of this increase is rather intimately linked with carburetor design, but the general laws which govern it are known.

Returning again to the example given in connection with discussion of volumetric efficiency, assume that the fuel-flow into the cylinder is governed by the same pressure difference as the air-

flow. As before, when the atmospheric pressure is P_1, let the pressure difference be $(P_1 - \frac{P_1}{m})$ and when the atmospheric pressure is P_3, let the pressure difference be $(P_3 - \frac{P_3}{m})$. Elementary mathematics shows that the pressure differences are in the same ratio as the atmospheric pressures. Since a change in atmospheric pressure does not appreciably change fuel density,

FIG. 8.

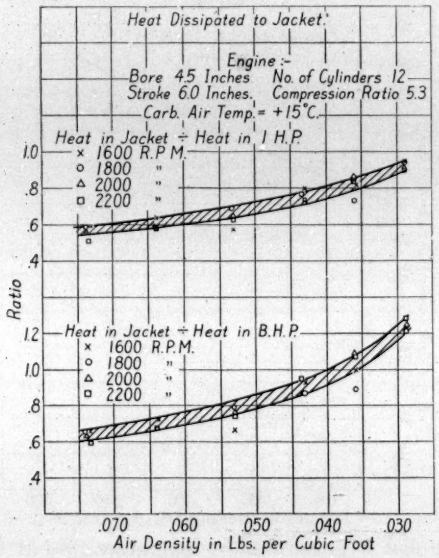


and since flow is proportional to the square root of the head, fuel-flow may be said to vary approximately as the square root of the pressure. Air-flow by weight has already been shown to decrease as the pressure. It follows, therefore, that the mixture becomes richer as the pressure decreases.

In the case of a difference in air temperature the pressure difference and hence the fuel-flow remains constant while the weight of air-flow has already been shown to vary inversely approximately as the square root of the absolute temperature. The decrease in temperature with altitude thus tends to lean the mixture.

Fig. 8 shows the percentages of enrichment with altitude based on the relations just given. The leaning of the mixture due to a decrease in temperature is somewhat greater than shown, as ordinarily a decrease in fuel temperature accompanies a decrease in air temperature. A decrease in fuel temperature produces an increase in viscosity which tends to reduce the flow still further.

FIG. 9.



While the amount of this reduction is very dependent upon the design of the fuel orifice, it usually is great enough to far outbalance the extremely small increase in weight of flow resulting from the slight increase in fuel density. In connection with Fig. 8 it should be pointed out that present-day carburetors are provided with manual adjustments which enable the pilot to alter the fuel-air ratio as he desires. Hence Fig. 8 indicates not the enrichment that ordinarily does take place, but rather what would take place were a mixture ratio control not provided or used.

As the amount of power developed by an engine decreases, the ratio between the heat dissipated to the jacket water and the horsepower increases. Hence this ratio increases with altitude. Fig. 9 shows one example of such an increase. The agreement between tests of different engines is not sufficiently good to warrant plotting the probable variation with altitude. This is not particularly serious as the importance of this ratio rests upon the fact that it dictates radiator requirements, and service tests have shown that for a conventional engine (not supercharged), a radiator which provides adequate cooling at sea-level is more than adequate at altitude. It is well to remember in connection with this subject that heat is dissipated to the jacket during the exhaust stroke as well as during the working stroke. Were this not so, it would be difficult to reconcile Fig. 9 with the fact that indicated thermal efficiency is unaffected by altitude.

Thus far discussion has been confined to *changes* in performance resulting from a *change* of altitude. It is now in order to consider what determines the performance at a given altitude.

First of all, mention should be made of one characteristic difference between aviation and automobile engines. In the automobile engine there is no definite relation between speed and load. Full load may be required at any speed. This is not true of the aviation engine whose power is delivered to a propeller. The power and torque required to drive a propeller are very nearly proportional to the cube and square of the speed, respectively. Tables I and II have been calculated on the assumption that the cube and square relations are exact, and they show the percentages of full-load power and torque necessary to drive a propeller at various speeds. Each vertical column is based upon the full-load speed corresponding to the 100 per cent. value at the head of the column.

It is common knowledge that the higher power of aviation engines is due in large measure to high volumetric efficiencies. High volumetric efficiency is the result of the proper choice and combination of valve timing, valve area, intake and exhaust pipe size, etc., and the combination which gives good results at one speed may give very poor results at another speed. This is a serious handicap to the designer of an automobile engine, but no handicap at all to the aviation engine designer who seeks high volumetric efficiency at one speed only.

TABLE I.

R. P. M.	Per Cent. of Maximum Power.													
2500	100													
2400	88.4	100												
2300	77.8	88.0	100											
2200	68.1	76.9	87.5	100										
2100	59.3	67.0	76.2	87.1	100									
2000	51.2	57.9	65.8	75.2	86.3	100								
1900	43.9	49.6	56.4	64.5	74.0	85.7	100							
1800	37.3	42.2	47.9	54.8	62.9	72.8	85.0	100						
1700	31.4	35.6	40.4	46.2	53.0	61.4	71.6	84.2	100					
1600	26.2	29.6	33.7	38.5	44.2	51.2	59.8	70.2	83.4	100				
1500	21.6	24.4	27.7	31.7	36.4	42.2	49.2	57.9	68.7	82.4	100			
1400	17.6	19.8	22.6	25.8	29.6	34.3	40.0	47.1	55.8	67.0	81.3	100		
1300	14.0	15.9	18.0	20.6	23.7	27.4	32.0	37.7	44.7	53.6	65.1	80.0	100	
1200	11.1	12.5	14.2	16.2	18.6	21.6	25.2	29.6	35.2	42.2	51.2	63.0	78.6	100
1100	8.5	9.6	10.9	12.5	14.4	16.6	19.4	22.8	27.1	32.5	39.4	48.5	60.6	77.0
1000	6.4	7.2	8.2	9.4	10.8	12.5	14.6	17.1	20.3	24.4	29.6	36.4	45.5	57.8
900	4.7	5.3	6.0	6.9	7.9	9.1	10.6	12.5	14.8	17.8	21.6	26.6	33.2	42.2
800	3.3	3.7	4.2	4.8	5.5	6.4	7.5	8.8	10.4	12.5	15.2	18.7	23.3	29.6
700	2.2	2.5	2.8	3.2	3.7	4.3	5.0	5.9	7.0	8.4	10.2	12.5	15.6	19.8
600	1.4	1.6	1.8	2.0	2.3	2.7	3.2	3.7	4.4	5.3	6.4	7.9	9.8	12.5
500	.8	.9	1.0	1.2	1.3	1.6	1.8	2.1	2.5	3.1	3.7	4.6	5.7	7.2
400	.4	.4	.5	.6	.7	.8	.9	1.1	1.3	1.6	1.9	2.3	2.9	3.7
300	.2	.2	.2	.3	.3	.3	.4	.5	.6	.7	.8	1.0	1.2	1.6
200	.1	.06	.06	.08	.09	.1	.1	.1	.2	.2	.2	.3	.4	.5
100	.006	.007	.008	.009	.01	.01	.01	.02	.02	.02	.03	.04	.05	.06

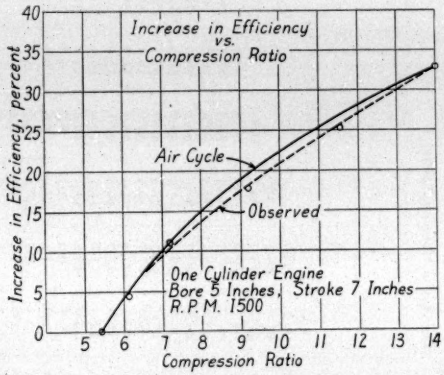
TABLE II.

R. P. M.	Per Cent. of Maximum Torque.													
2500	100													
2400	92.2	100												
2300	84.6	91.8	100											
2200	77.4	84.0	91.4	100										
2100	70.6	76.6	83.4	91.2	100									
2000	64.0	69.4	75.6	82.6	90.8	100								
1900	57.8	62.6	68.2	74.6	81.9	90.2	100							
1800	51.8	56.2	61.2	67.0	73.5	81.0	89.8	100						
1700	46.2	50.2	54.6	59.7	65.6	72.2	80.0	89.2	100					
1600	41.0	44.4	48.4	52.9	58.0	64.0	70.9	79.0	88.6	100				
1500	36.0	39.1	42.5	46.5	50.5	56.2	62.3	69.4	77.8	87.9	100			
1400	31.4	34.0	37.0	40.5	44.5	49.0	54.3	60.5	67.8	76.6	87.1	100		
1300	27.0	29.3	32.0	34.9	38.3	42.3	46.8	52.2	58.5	66.0	75.1	86.2	100	
1200	23.0	25.0	27.2	29.8	32.7	36.0	39.9	44.4	49.8	56.3	64.0	73.5	85.2	100
1100	19.4	21.0	22.9	25.0	27.4	30.2	33.5	37.3	41.9	47.3	53.8	61.8	71.6	84.0
1000	16.0	17.3	18.9	20.7	22.7	25.0	27.7	30.9	34.6	39.1	44.4	51.0	59.2	69.4
900	13.0	14.1	15.3	16.7	18.4	20.3	22.4	25.0	28.0	31.7	36.0	41.3	47.9	56.2
800	10.2	11.1	12.1	13.2	14.5	16.0	17.7	19.8	22.1	25.0	28.4	32.6	37.9	44.4
700	7.8	8.5	9.3	10.1	11.1	12.3	13.6	15.1	17.0	19.1	21.8	25.0	29.0	34.0
600	5.8	6.2	6.8	7.4	8.2	9.0	10.0	11.1	12.5	14.1	16.0	18.4	21.3	25.0
500	4.0	4.3	4.7	5.2	5.7	6.3	6.9	7.7	8.7	9.8	11.1	12.8	14.8	17.4
400	2.6	2.8	3.0	3.3	3.6	4.0	4.4	4.9	5.6	6.2	7.1	8.2	9.5	11.1
300	1.4	1.6	1.7	1.9	2.0	2.3	2.5	2.8	3.1	3.5	4.0	4.6	5.3	6.3
200	.6	.7	.8	.8	.9	1.0	1.1	1.2	1.4	1.6	1.8	2.0	2.4	2.8
100	.2	.2	.2	.2	.2	.2	.3	.3	.3	.4	.4	.5	.6	.7

The influence of air temperature upon volumetric efficiency has already been discussed. Jacket water temperature also has a slight influence which is probably due for the most part to the fact that intake manifolds are water-jacketed. A decrease in jacket water temperature always tends to increase indicated horsepower, but as it also increases friction, the resultant effect on brake horsepower is uncertain.

The theoretical efficiency of an engine operating on the Otto

FIG. 10.



cycle is given by the equation $E = 1 - \frac{1}{R^{n-1}}$, where E is the efficiency, R the expansion ratio, and n the ratio of the specific heat at constant pressure to the specific heat at constant volume. This efficiency, calculated for air as the working medium for which n is approximately 1.4, is a common standard for judging internal combustion engine performance, although there are theoretical as well as practical reasons why this efficiency can not be attained. Experiments at the Bureau of Standards indicate that under normal operating conditions—that is with no pre-ignition or detonation present—an engine's efficiency at one expansion ratio is the same percentage of the air-cycle efficiency that it is at another ratio. Hence, if its efficiency at one ratio be known, its efficiency at other ratios can be predicted with reasonable certainty. This will be evident from Fig. 10, which shows the results

of a group of tests made to ascertain what improvement in the performance of an engine would result from increasing its expansion ratio above its normal value of 5.3.

An increase in thermal efficiency means an increase in the amount of work obtained from unit weight of charge, and, as neither volumetric efficiency nor air density is affected to any extent by expansion ratio, it produces an increase in maximum engine power, as well as a decrease in specific fuel consumption. From Fig. 10 it will be evident that an engine developing 400 horsepower with an expansion ratio of 5.3 would develop 532 horsepower with an expansion ratio of 14, provided the same fuel were used at both ratios and there was no serious detonation or pre-ignition at either ratio.

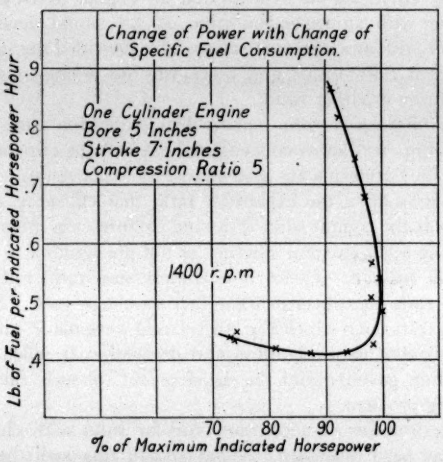
It is *compression ratio* rather than expansion ratio that is plotted in Fig. 10, but in conventional engines the two ratios are equal and the two terms are used almost interchangeably. Strictly speaking, it is upon the expansion ratio that efficiency depends, whereas it is the compression ratio and compression pressure that determine to a large extent whether or not pre-ignition or detonation will be present. Hence it is compression ratio rather than expansion ratio that dictates what fuel should be used. Needless to say, the tests upon which Fig. 10 is based were made with a fuel whose resistance to pre-ignition and detonation is much greater than aviation gasoline and the development of such fuels is an outstanding problem.

The desirability of a constant fuel-air ratio with change of altitude has been discussed. What should this ratio be? The answer will depend upon whether maximum power or minimum specific fuel consumption is desired. A majority of the aviation engines tested at the bureau have developed maximum power with ratios of between .07 and .08 pound of gasoline per pound of air. A fuel content between 15 per cent. and 20 per cent. less has given best economy. Theoretical considerations, which will not be discussed here, show why, if distribution and mixing were perfect, the chemical combining ratio of fuel to air would give neither maximum power nor minimum specific fuel consumption. Perfection not having been attained in either respect, an excess of fuel is desirable to ensure complete utilization of the air, which is the requirement for maximum power, and an excess of air to ensure

complete utilization of the fuel, which is the requirement for best economy.

What appears to be a bit of rather useful information is given in Fig. 11, namely, that when the fuel-air ratio is reduced until the power is 5 per cent. less than maximum, very nearly minimum specific fuel consumption is obtained. The value of this information rests upon the remarkable agreement in this respect shown

FIG. 11.



by tests of many engines under many conditions. The 5 per cent. value has no decided advantage over 6 per cent. or 4 per cent., nor is the specific fuel consumption ordinarily the absolute minimum. The important fact is that decreasing the power another 5 per cent. or 10 per cent. yields a rather trifling gain in efficiency and greatly increases the likelihood of backfiring in the carburetor and its attendant danger.

The above information should form a basis upon which an engine designer can estimate how much larger an engine should be for a given power output if it is to be operated for high efficiency rather than for maximum power. It also serves as a convenient basis for adjusting the fuel-air ratio in service for high

efficiency. This 5 per cent. value is only approximate and can be considered as applicable to brake horsepower as well as indicated horsepower. As a matter of fact, if an engine has a full-load mechanical efficiency of 90 per cent. when 95 per cent. of maximum indicated horsepower is obtained, the brake horsepower will be 94.4 per cent. of its maximum.

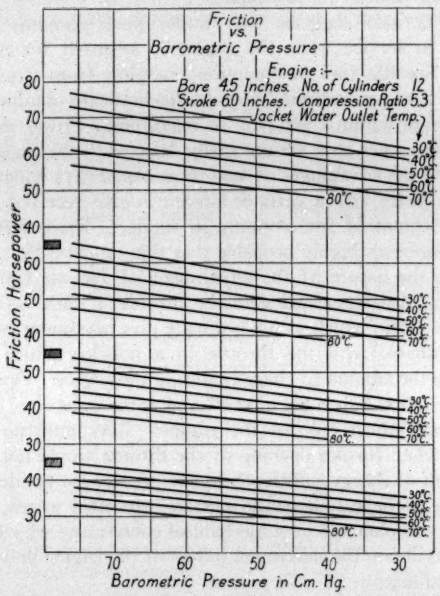
Fig. 11 also suggests why rather poor economy is often obtained in service when the engine is adjusted for maximum power. Specific fuel consumptions ranging from 0.44 to 0.57 pound of fuel per indicated horsepower hour are obtained with a power output within 1 per cent. of maximum. Power differences smaller than 1 per cent. are not easily detected by the engine operator nor are the advantages of lean mixtures always understood.

Some years ago a certain foreign engine received a rather enviable reputation for economy in service. From tests of the engine it appears highly probable that this economy was due primarily to the nature of the adjustment of fuel-air ratio. This adjustment was connected with the throttle in such fashion that the fuel-air ratio which would normally give maximum power was only obtainable with the throttle in a position which reduced materially the amount of charge entering the engine. Opening the throttle further, leaned the mixture and at the same time increased the amount which reached the engine. Maximum power was obtained when further opening of the throttle would not increase the amount of charge sufficiently to compensate for the decrease in power due to the leaning of the mixture. In other words, the high economy was obtained by a mechanical contrivance which made it possible to obtain the maximum power of the engine only with an economical mixture.

It has already been pointed out that knowledge of the factors governing engine friction is very incomplete. As to the magnitude of engine friction little can be said other than that it has been reduced to a point where mechanical efficiencies of over 90 per cent. at sea-level are not uncommon. An increase in engine speed produces a more than proportionate increase in friction and the tendency is thus for mechanical efficiency to decrease with speed. Piston friction constitutes a high percentage of the total, as will be evident from Fig. 12, which shows large changes in friction to have resulted from changes in jacket-water temperature. This

is, of course, a consequence of the influence of jacket-water temperature upon the viscosity of the oil upon the cylinder wall. Were an engine always to be operated with a low jacket-water temperature, oil of lower initial viscosity could be used and the

FIG. 12.



friction would not necessarily be greater than that of an engine whose normal operating temperature was much higher.

This discussion has dealt with what is to be expected. Experience shows that the unexpected still happens with disagreeable frequency. To attempt to explain why and to enumerate the many difficulties involved in engine testing would extend the discussion to an intolerable length. Suffice it to say, that improvements in reliability have greatly reduced the difference between predicted and actual performance.

THE RATE OF VISUAL WORK ON ALTERNATING FIELDS OF DIFFERENT BRIGHTNESSES.*

BY

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IN THE course of ordinary industrial tasks, the eyes are required to distinguish detail in fields where the intensity of illumination frequently varies greatly from place to place. Even in the simplest sort of visual work the eyes are constantly in motion, focussing and adapting themselves to new conditions. If the requirements of the work are such that the eyes must shift from bright to less bright areas, or *vice versa*, it is obvious that such conditions impose a handicap on quick and clear seeing.

The effectiveness of lighting is not necessarily determined by the amount of light flux falling upon or reflected by an object, but rather by the ease with which the object is seen. It has been well established, assuming a proper distribution of light for the particular work to be done, that the amount of visual work that can be accomplished per unit time varies with the intensity of illumination. These ideal conditions are often lacking in practical installations, and therefore experimental work has been undertaken to determine, quantitatively, the effect of variable intensities of illumination within the field of vision upon the rate with which visual work can be accomplished. The rate of visual work was chosen as a criterion of the effectiveness of the illumination since it is a resultant of all the factors influencing vision, assuming that factors other than visual are eliminated or reduced to the second order of importance.

Vision on alternating light and dark fields is subject to all the influences affecting vision in general; and of these, the factors of intensity of illumination and the adaptability of the eye to changing intensities must be considered as of primary importance. These essential factors may be regarded as working in opposition to each other since intensity can be made a factor in aiding vision

* Communicated by the Director of the Laboratory.

while the adaptation-factor is a handicap that must be overcome. There are data available on each of these factors separately, but they are of little use in calculations of lighting problems where the relative effect of uniform illumination is to be compared with illumination of variable intensities and its corresponding differences in brightness and changes in adaptation of the eyes. Moreover, the problem differs from the usual adaptation experiments in that complete adaptation is not required; but rather a partial one that takes place until an image is obtained of sufficient clearness to be recognized.

The visual task selected was the comparison of two letters appearing in a white field, and separated by approximately eight inches. By operating a key the subject could change both letters and was instructed to note and record, by pressing a counting device, each time the two were alike. The number of times the letters were changed per unit time was taken as a measure of the amount of visual work done and the accuracy with which the coincidences were recorded was considered a check on the attentiveness of the subject to the work.

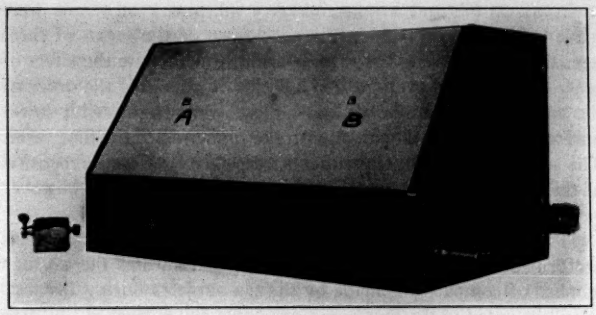
The relation of brightness or intensity of illumination to the rate of doing visual work was obtained by illuminating the field of the apparatus to a uniform brightness and measuring the work accomplished in a given time. The experiment was then repeated as before, excepting that the field was rather sharply divided equally between light and dark areas, so that in performing the task the eyes were required to alternate between two fields of different brightnesses and, therefore, to re-adapt themselves to each new condition. These data served as a means of evaluating the effect of adaptation in terms of intensity.

Fig. 1 illustrates the apparatus used. The typewritten letters appeared in the openings *A* and *B* of the flat white surface of the apparatus. Directly below one opening was mounted a drum upon which 100 letters were arranged in shuffled order, ten different letters being used. A second drum was placed under the other opening with 101 letters in a different shuffled order, using the same ten letters as before. Two ring gears of 100 and 101 teeth were mounted on the respective drums, the two gears meshing with an idler. A pawl, acting on the teeth of the idler, thus turned each drum through the space occupied by one letter each

time the key was operated. Such an arrangement therefore provided a series of 10,100 pairs of letters before repetition occurred, and since the same ten letters were used on both drums, 1010 coincidences were obtained. The subject was instructed to operate the key as fast as was consistent with accuracy.

The experimental work was conducted in two parts: The first was a series of three orders of contrast between the two halves of the field, and the second part of a single contrast which, on account of its difficulty due to low values of brightness, was done separately. The first series of contrasts, 20 ml. and 5 ml.,[1] 20 ml. and

FIG. 1.



View of apparatus used in experiment.

1 ml., and 5 ml. and 1 ml., respectively, was chosen as representing the range of intensity that would be of the most use in present-day industrial lighting. The single order of contrast, 1.0 ml. and 0.05 ml., was made as low as possible, the visibility of the test-objects being the limiting factor, to extend the application of the data to such conditions of high contrast and low brightnesses as are common in street-lighting practice.

The apparatus was used in a room where the subject was as free as possible from external disturbance. The first series of the experiments began with the surface of the apparatus illumi-

[1] The brightness of a reflecting surface is the effect produced by illumination. A reflecting surface of 100 per cent. reflection-factor (a perfect white) under an illumination of one foot-candle has an average brightness of 1.076 millilamberts. The average brightness in millilamberts of *any* surface is equal to 1.076 × foot-candles × reflection-factor.

nated to a uniform brightness of 20 ml. for the background. At a signal the subject began work and continued for fifteen minutes, the experiment stopped and the counters indicating the number of trials and the number of coincidences were read. The lighting was immediately changed, so that one-half of the field was illuminated so as to provide a brightness of background of 20 ml. and the other half illuminated to 5 ml., and the experiment repeated as before. Finally the experiment was repeated, using a uniform illumination at 5 ml. Thus the actual working time for each run was forty-five minutes, and one of these runs was made each day by each subject. On subsequent days with contrasts of 20 ml. and 1 ml., and of 5 ml. and 1 ml., runs were made in the same way. At the conclusion of the series of three tests they were repeated in reverse order, to eliminate the effect of practice in the final average. In addition to changing the order of the fourth, fifth and sixth runs, these were begun with lower uniform intensity rather than with the higher as in the first three. The seventh to twelfth experiments, respectively, were a repetition of the first six. In all, ten observers were used, and each observer gave four sets of data for each condition of illumination.

The data are summarized in the table for each experiment performed. The second column (all data) indicates the average number of comparisons made by the ten subjects during the test. It will be noted in the first experiment that the change in uniform brightness from 20 ml. to 5 ml. did not materially alter the amount of work done in the time allowed. Such a result might have been anticipated since the test-object was of black ink on white paper, read at the usual reading distance, and illuminated to 5 ml., which made its visibility very good. However, when the two halves of the field were illuminated to 20 ml. and 5 ml., respectively, the change in the amount of work done is significant when the probable error of the measurements is considered. This decrease in the amount of work done must be ascribed to the additional time the eyes required for adapting to the alternating intensities since the mere change in intensity would not account for such a decrease.

The second of the three experiments is that in which the greatest effect in the reduction of the working rate would be expected. A decrease of 11 per cent. was obtained in alternating from a field

of 20 ml. to one of 1 ml. as compared with the uniform illumination of 20 ml. This change is quite significant since it is something like seven times its probable error. As before, when the illumination was made uniform, even if considerably less than the average of the high and the low intensities, there was a slight increase in the work done over that done under an alternation of the two. But in this case, intensity of illumination is a factor, as the work done under 1 ml. was distinctly less than under 20 ml., and therefore the decrease in production rate cannot be ascribed entirely to contrast, although an examination of the results will indicate that adaptation does play an important part. In other words, all the gain in speed that high intensity would give to the subject in recognizing the letter on one side of the field would more than be offset by the time the eyes require for adapting themselves to the lower intensity on the other side. It must be concluded that a uniform brightness of 1 ml. would produce better results than an average of 10.5 ml., but consisting of comparatively bright and dark areas, such as 20 ml. and 1 ml.

The third range of contrast, 5 ml. and 1 ml., produced about the same results as in the preceding experiments. The disagreement in the fact that uniform intensity of 1 ml. did not produce better results than the combination of 5 ml. and 1 ml. is of little importance, as the differences shown by the data are but slightly in excess of the probable error of the measurements.

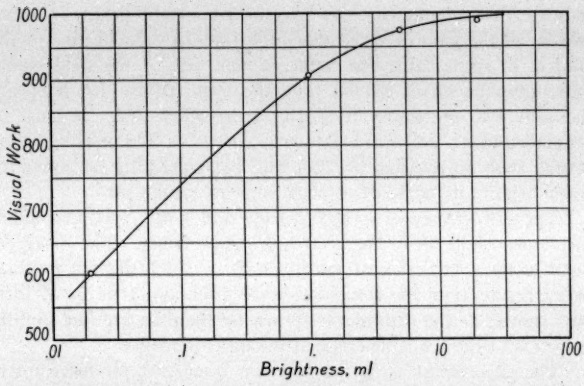
The experiment using the very low intensities produced interesting results as the changes in production rate were quite marked. The alternating of the two intensities cut the production rate to sensibly the same as with the lowest uniform intensity. From the results of this experiment it may be concluded that the slowing up in speed due to the presence of high contrasts takes place over ranges of intensity from threshold to very high values.

The method is strictly one of comparison of the effects produced by two systems of illumination on the amount of visual work done in a given time, so it is essential that all parts of each experiment be performed with the same amount of precision. It will be seen from the table of data that the average precision for the three parts of each experiment were approximately the same. In the second column (5 per cent. limits) is a recapitulation of the data excluding all experiments in which there was a difference

of more than 5 per cent. in the accuracy of the work done between the three parts.

This test, while not designed to show the effect of intensity of illumination on the production rate of visual work, provides a very practical example of the results that can be obtained with good lighting. It will be noted in Fig. 2 that the amount of work done in each period is always greater for the greater brightness and, considering the reliability of the data, shows a positive argument for higher intensities of illumination than are common.

FIG. 2.



Showing the relation of brightness of background (or intensity of illumination) to the amount of work done when production depends appreciably upon the speed and accuracy of vision. The field was of uniform brightness.

While it is not generally feasible to obtain data in the laboratory that will solve directly many of the problems encountered in lighting practice, these results do show what may be expected in practice, when the fundamental factors involved are the same as those considered in the laboratory work. Furthermore, such investigations supply foundational data upon which to build an art of lighting for effective vision and efficient work. The results of the experiment as a whole may be briefly stated as:

That an abundance of light, *if improperly placed,* can be a handicap to vision, and again emphasizes the importance of designing industrial and office lighting to meet the needs of the individual job.

That the "average value" of illumination over a certain area of factory or office may be a very misleading figure in denoting the effectiveness of the illumination.

That high contrasts in brightness in the important visual field reduces the "speed of vision" and consequently slows up production when the latter depends appreciably upon quick and accurate seeing.

While it was to be expected that the continual and rapid re-adapting of the eyes to different brightnesses would be fatiguing, it was of interest to learn that the magnitude of the eye-fatigue was such as to cause the observers to voluntarily state that the alternating condition was much more severe than that of the uniform brightness.

SUMMARY OF DATA.

TABLE I.*

Brightness of Field in ml.		Average Working Rate.		Per Cent. Change.	Per Cent. Accuracy.	Mean Variation, Excluding Individual Differences.	Probable Error of Average Working Rate.
Left Half.	Right Half.	All Data.	5 % Limits.	All Data.	All Data.		
20	20	981	987	100.0	93.8	116	17.8
20	5	949	949	96.8	93.5	120	18.5
5	5	972	980	99.0	94.0	118	18.2
20	20	988	979	100.0	94.4	74	11.4
20	1	884	869	88.9	92.9	88	13.5
1	1	918	881	92.3	94.2	94	14.5
5	5	978	971	100.0	93.7	77	11.9
5	1	907	918	92.7	95.1	85	13.1
1	1	891	871	91.1	94.6	89	13.5

*Owing to the difficulty of the last test the duration of each test was reduced from fifteen minutes to ten minutes.

TABLE II.*

1	1	689	...	100.0	93.3	54.6	8.0
1	.05	429	...	62.3	92.6	41.8	6.2
.05	.05	394	...	57.2	90.3	45.7	6.7

*The shortening of the time and the practice gained in the previous tests can account for the small discrepancy in the working rate between the same conditions of lighting (1 ml.) shown in Tables I and II.

For each condition of lighting, the results of four tests on ten observers were averaged for the values given in the above tables.

Determination of Potassium as Acid Tartrate.—R. MAURICE describes in *Ann. Chim. Anal., etc.* ([2], 1925, 7, 161), a procedure in which advantage is taken of the sparing solubility of acid potassium tartrate, but instead of weighing the precipitate, it is titrated with standard acid after over-neutralizing it with standard alkali. The precipitant is acid sodium tartrate dissolved in a mixture of equal parts of pure methanol and water. Methanol of high purity and almost absolute can now be obtained in abundance and at reasonable rates. If sulphates are present, sodium sulphate will separate, but as this is neutral, it will not interfere with the determination by titration. After obtaining the precipitate of acid potassium tartrate, and decanting the liquid through a suitable filter, it is washed with the methanol-water mixture until a few drops of the filtrate give no appreciable precipitate with a concentrated solution of potassium acetate. As is well known, the separation of acid potassium tartrate is promoted by stirring, which should not be omitted. It is advisable not to use more than 100 c.c. of the washing liquid, divided into as many small portions as possible. After washing, the filter and contents are placed in the beaker containing the rest of the precipitate, over-neutralized by standard sodium hydroxide, and the excess of alkali determined by titration with standard acid. Test analyses with samples containing sulphates, nitrates and chlorides, as well as magnesium compounds, showed approximations to about 0.5 per cent. of the amount present. To insure the complete precipitation of the acid potassium tartrate, it is necessary to stir the liquid well and even to allow it to stand over night. The procedure is recommended for the analysis of fertilizers. H. L.

Some Geophysical Problems. R. L. FARIS. (*Jour. Wash. Acad. Sci.*, Aug. 19, 1925.)—We speak of elevation above the sea so glibly that it comes as something of a shock to learn that mean sea-level is a hard thing to determine. "As a geophysical problem it is found to have variations from year to year, and through a period of years. Neither are all of its variations yet known nor are their causes understood in detail. So it is yet a problem of prime importance for systematic observation and study."

At the 1924 meeting of the International Geodetic and Geophysical Union in Madrid the Hayford Spheroid was officially adopted, and the validity of the theory of isostasy was recognized.

Doctor Meinesz, of the Dutch Geodetic Commission, has measured the acceleration of gravity on a submerged submarine, thus opening the way for other determinations on the oceans for which there is great need. There is lack of a gravity instrument that shall be easier to use in regard to the labor both of observation and of computation. It is held by some that the continents are slowly drifting like huge rafts. To find whether this is so more latitude and longitude determinations are required. G. F. S.

X-RAY ANALYSIS OF SOME MIXED CRYSTALS OF THE SILVER HALIDES.*

BY

R. B. WILSEY.

As a result of equilibrium and potential measurements on mixed precipitates of the silver halides, Thiel [1] concludes that silver iodide and silver bromide are able to form homogeneous mixtures with as high as 30 molecular per cent. of silver iodide; Trivelli and Sheppard [2] also found that silver iodide formed homogeneous mixtures with silver bromide. The mixed crystals exhibited double refraction, whereas crystals of pure silver bromide did not. This fact led Trivelli [3] to consider that the additional speed obtainable in photographic emulsions of silver bromide containing small amounts of silver iodide might be associated with a state of strain set up by the presence of silver iodide incorporated in the silver bromide lattice; the strain was caused by a distortion or stretching of the silver bromide lattice by the included silver iodide. In order to find what occurred in the lattice structure, Mr. Trivelli suggested that such mixtures be studied by the methods of X-ray crystal analysis.

The powdered crystal method of Hull [4] was used. A survey was made of the structures of mixtures varying from pure silver bromide to pure silver iodide. Most of the mixtures were prepared by fusion; in one set the samples were cooled suddenly by quenching in water, and in another set the samples were allowed to cool more slowly by standing in the room; accurate control of the rate of cooling was not attempted. A few samples, containing up to 30 molecular per cent. of silver iodide, were prepared by precipitation, by adding silver nitrate to a mixed solution of potassium bromide and potassium iodide.

The apparatus for the X-ray analysis was that previously used for the study of the structures of pure silver halides.[5] Silver chloride and silver bromide had been found to crystallize in the Na-Cl (simple cubic) type of lattice, while silver iodide crystal-

* Communicated by Dr. C. E. K. Mees, Director of Laboratory and Associate Editor of this JOURNAL. Published as communication No. 242 from the Research Laboratory of the Eastman Kodak Company.

lized in two forms, the Zn-S type (diamond cubic) and the Zn-O type (hexagonal). The two forms of silver iodide are not much different; in either form, each atom of one kind is surrounded by four atoms of the other kind at the corners of a tetrahedron.

Since silver bromide and silver iodide crystallize in different forms, it is of special interest to find what occurs when the two salts are fused together. At small concentrations of silver iodide, the crystals were found to be simple cubic, as pure silver bromide, but with a slightly enlarged lattice spacing. After sufficient addition of silver iodide, the stronger lines of its own characteristic diffraction began to appear, with a slightly lower lattice spacing than that of pure silver iodide, indicating that a small amount of silver bromide had been taken into the silver iodide structure. It is not possible from the present data to state at what point the silver iodide crystal begins to form separately; it was found that in a simple mixture of the two powdered crystals, about 10 per cent. of silver iodide had to be present to indicate its presence by any lines of its own pattern.

With further addition of silver iodide, more of it crystallized separately from the silver bromide structure; but portions of it continued to unite with the silver bromide until the maximum stretching of the lattice spacing occurred when the total mixture contained about 70 molecular per cent. of silver iodide. With further additions of silver iodide, the proportion uniting with the silver bromide decreased, as shown by the lowered lattice spacing, and the silver iodide structure predominated. The lattice constants observed in the various mixtures are recorded in Table I. The d_{100} spacing is given for the simple cubic AgBr lattice, and the side of the elementary diamond type cube is given for the AgI structure; the silver iodide crystals usually appeared in the diffraction patterns as mixtures of the cubic and hexagonal forms; to simplify the table, only the cubic dimensions are recorded; the side of the corresponding hexagonal prism may be found by dividing the cube edge by $\sqrt{2}$.

It is apparent that the lattice spacings of the mixed crystals are affected by the conditions under which the crystal is formed, which, of course, means that the relative proportions of the components of the mixed crystal vary according to the conditions of formation. The size of the simple cubic structure in the quenched mixture generally exceeds that of the sample cooled slowly; the

difference cannot be attributed to errors in measurement, which lie within 0.1 per cent. except in those cases where the small number of lines appearing makes the value more uncertain. In the quenched mixture, the silver iodide structure first shows up at 60 per cent. AgI, whereas in the samples cooled slowly the iodide structure appears at 40 per cent. AgI; this is further evidence that in the quenched mixtures more of the iodide goes into the silver bromide structure than in the cases of the samples which are cooled slowly.

The variations in lattice constant from one concentration to

TABLE I.

Mol. per cent. AgI.	Mol. per cent. AgBr.	Lattice Constants in Ångströms.				
		Fused and Cooled Slowly.		Fused and Quenched.		Precipitated.
		Simple Cubic.	Diamond Cubic.	Simple Cubic.	Diamond Cubic.	Simple Cubic.
0	100	2.884		2.884		2.884
10	90	2.907		2.908		2.903
20	80	2.921		2.927		2.920
30	70	2.930		2.938		2.939
40	60	2.948	6.47*	2.954		
50	50	2.956	6.47*	2.966		
60	40	2.959	6.47	2.980	6.48*	
70	30	3.007 2.973 2.997	6.48	2.978	6.48	
80	20	2.958	6.47	2.948*	6.48*	
90	10		6.472	2.949*	6.483	
95	5		6.481		6.487	
100	0		6.493		6.493	

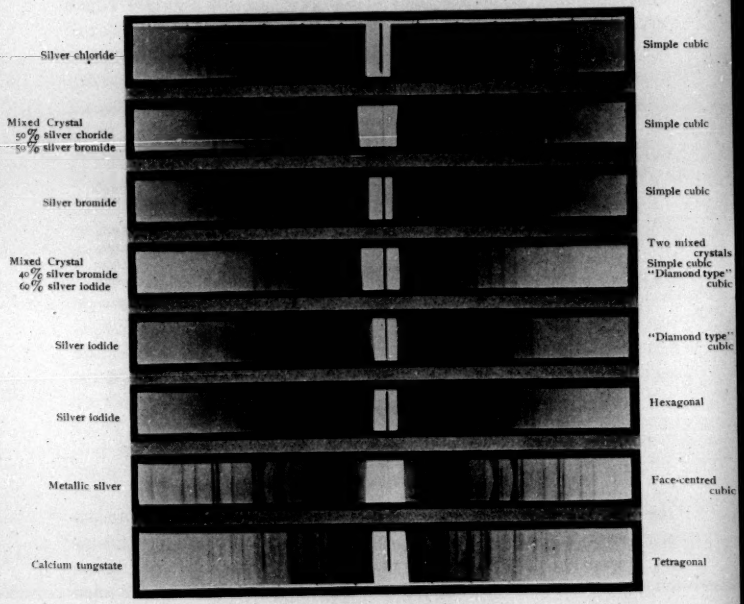
* Patterns were so faint as to show only three of the strongest lines.

the next show irregularities not attributable to experimental measurements, but which must be due to variations in the conditions of formation of the mixed crystals. In the case of the fusion of 70 per cent. AgI with 30 per cent. AgBr which was cooled slowly, the three values given were for samples taken from different parts of the same crucible, and show a maximum variation of 1 per cent. In order to obtain consistent results, it is evident that the fused mixture would have to be cooled very slowly under carefully controlled conditions. This was not considered necessary for the purposes of the present survey. The precipitated mixtures examined (10, 20, and 30 molecular per cent. AgI) gave spacings agreeing well with those of the fused mixtures.

A special experimental procedure would be required to estimate the relative quantities of the two crystal forms in the various samples, and to find the amount of each component in each of the two mixed crystals. This was not attempted in the present experiments.

Following the conception of atomic " radii " in crystals as sug-

FIG. 1.



X-ray crystal structure diffraction patterns.

gested by Bragg [6] and later modified by Davey,[7] the relative proportions of silver bromide and silver iodide in the mixed crystal may be calculated from the lattice constant of the mixed crystal and that of pure silver bromide; assuming Davey's value of 0.25° Ångström for the difference between the atomic radii of bromine and iodine, the maximum observed spacing for the mixed

crystal (3.007 Å.) would correspond to practically equal numbers of bromine and iodine atoms in the structure; this proportion of iodine atoms occurs only in the presence of a large excess of silver iodide. On the same assumption, when 10 per cent. of silver iodide is fused with silver bromide, practically all of it goes into the silver bromide lattice. On fusing 30 molecular per cent. AgI with 70 molecular per cent. AgBr, the calculation indicates that the simple cubic crystal contains 22 parts of AgI to 78 parts of AgBr; however, the quantity of silver iodide crystallizing in its own form was insufficient to show its presence in the photograph.

In the diffraction patterns of the mixed crystals, the lines are as sharp as in the patterns of the pure crystals, and all the spacings are enlarged in the same ratio; so far as the X-ray method can distinguish, the mixed crystals have as perfect a geometric arrangement of the atoms as the pure crystals. Diffraction photographs of samples of mixed crystals and pure crystals are shown in Fig. 1.

It seems quite plausible that the enlargement of the silver bromide lattice by the incorporation of silver iodide, and the resulting changes in the interatomic forces may be closely connected with the increased speed obtained in photographic emulsions. Trivelli's theory as to the effect of the silver iodide upon the silver bromide lattice has been verified. How the enlargement of the lattice operates to increase photographic speed is not yet understood. The effect is not proportional to the enlargement of the lattice, as the proportions of iodide giving the best effect upon photographic speed are small, being of the order of 5 per cent.

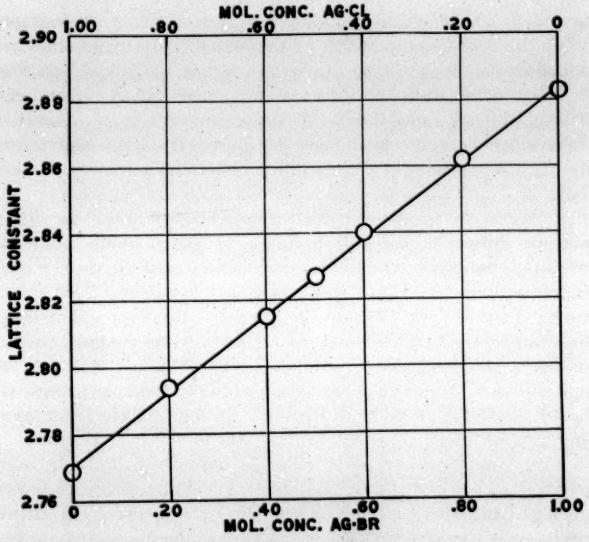
Fused mixtures of silver chloride and silver bromide were investigated and found to have simple cubic structures, with lattice spacings lying between those of the pure components. The lattice spacing was found to be a linear function of the molecular concentration of each component, as shown by the curve in Fig. 2; in no case does the observed spacing depart from the straight line by as much as 0.1 per cent. The lattice spacings of the various fused mixtures of silver chloride and silver bromide are recorded in Table II.

The equilibrium between silver chloride and silver bromide when precipitated together from mixed solutions of potassium chloride and potassium bromide has been studied by Küster;[8] he concluded that the two salts are present in the precipitate in the

form of an isomorphous mixture. Thiel [1] concluded that silver chloride and silver bromide form homogeneous mixtures in all proportions of the components; these conclusions are now confirmed by the X-ray method.

The first X-ray analysis of mixed crystals known to the writer is that of Vegard,[9] who found that a mixed crystal of equal molecular concentrations of potassium chloride and potassium

FIG. 2.



bromide had the same simple cubic structure as either of the two components, but with a lattice spacing midway between those of the pure salts. Similar results were obtained with mixtures of the chlorides of ammonium and potassium, and mixtures of the sulphates of ammonium and potassium. In these mixed crystals Vegard found that the lattice spacing could be computed from the spacings of the pure components by assuming a linear relation between lattice spacing and molecular concentration, as was found in the present experiments with mixtures of silver chloride and

silver bromide. Such a linear relation has also been found by various investigators in the cases of some metallic alloys, whereas other combinations of metals gave quite different effects depending on the characteristics of the component metals.

SUMMARY.

Mixed crystals of silver chloride with silver bromide, and silver bromide with silver iodide, in various proportions of the components, were examined by the X-ray powder method of crystal structure analysis. The AgCl-AgBr mixtures showed the simple cubic structure (NaCl type) characteristic of the pure components, with a lattice spacing lying between those of the two components and following a linear relation with the molecular concentration of each component.

Most of the AgBr-AgI mixtures showed the existence of two

TABLE II.

Mol. per cent. AgCl.	Mol. per cent. AgBr.	Lattice Spacing 100 Planes.
0	100	2.884
20	80	2.862
40	60	2.840
50	50	2.827
60	40	2.815
80	20	2.794
100	0	2.770

mixed crystals, one the simple cubic type characteristic of pure silver bromide, but with an enlarged lattice spacing, and the other the diamond cubic or hexagonal type of the silver iodide crystal, with a slightly diminished lattice constant. In mixtures up to and including 70 molecular per cent. of silver iodide, the simple cubic structure predominated, its lattice spacing increasing with the percentage of silver iodide fused in the mixture. Silver iodide appears to go into the silver bromide lattice quite readily, the calculations indicating that the largest simple cubic structure observed contained as much iodide as bromide; while judging by the effect upon the lattice constant of the silver iodide structure, very little silver bromide entered the silver iodide lattice.† Beyond 70 molecular per cent. of silver iodide its own structure

† Thiel reached the same conclusion; he estimated that about 5 per cent. of silver bromide could unite with silver iodide to form a homogeneous mixture.

predominated in the mixture, and the lattice spacing of the simple cubic structure diminished, owing to a smaller fraction of silver iodide entering the silver bromide lattice.

The prediction of Trivelli that the incorporation of silver iodide would be found to enlarge the silver bromide lattice was verified; and it seems very likely that this enlargement is connected in some way with the increased speed obtainable in photographic emulsions of silver bromide containing small percentages of silver iodide.

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Variation of the Surface Tension of Liquids under Radiation. A. GRUMBACH and S. SCHLIVITCH. (*Comptes Rendus*, Aug. 10, 1925.)—Very small changes in the height of a liquid in a capillary tube were observed to take place when radiation fell upon the meniscus, but measurable effects were first obtained by using an arrangement that has been employed to show the influence of a change of temperature on the capillary constant of the liquid heated. A liquid index was introduced into a horizontal tube wet by the liquid. At each end of the index is a meniscus curving into the liquid and striving to flatten itself. The index comes to rest under the action of its weight and of the forces exerted by these two terminal films, each of which tends to draw the index toward itself. If anything strengthens the surface tension of one of the films, the index moves toward it. When the light from a mercury lamp fell on one meniscus, the index moved toward that end, thus showing that the surface tension grew larger under illumination. Among the liquids for which this effect occurred were ordinary petroleum, xylene in alcoholic solution and fluoresceine dissolved in water or in alcohol. No liquids were discovered that gave the opposite action. In the case of petroleum the shift of position was .43 mm. in sixteen minutes. Fortunately for the simplicity of the experiment a rise of temperature lessens the surface tension, so that whatever elevation of temperature is caused by the radiation tends to make smaller the observed motion. G. F. S.

MAGNETIC FLUX ENERGY.*

BY

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CALCULATIONS involving electrical currents are almost always based on the conception of a "complete circuit" as advocated by Maxwell. The method advocated by Ampère, based on the actions of parts of current-carrying conductors on each other, has apparently received but little attention. Believing that the latter system was worthy of more consideration than it had received, the writer investigated the fundamental properties of the single, straight conductor, far removed from all others, believing that a unit length of such a conductor might lead to some much needed, more basic, fundamentals, as there is no such thing as a fundamental or unit "complete circuit," while such a unit length of single conductor is fundamental and embodies absolute ultimata.

In the February, 1925, issue of this JOURNAL (p. 235[1]), the writer described some of the results of this investigation in which, by a new, simple and direct proof, based only on a few well-established and unquestioned laws, and not involving any infinities, self-inductances, postulates, etc., the simple constant $i^2/2$ ergs per cm. was obtained for the stored energy in such a straight conductor. This is analogous to the $mv^2/2$ ergs stored in a moving body, and when the former is written $li^2/2$, in which l is the length of the conductor, or the measure of the pile of discs of flux, the two expressions are also alike in form, which may have some significance. This constant, if correct, is important as it is probably one of the most fundamental constants in electrodynamics, and it serves as the basis of a number of important deductions.

But when this same energy is calculated by the older methods, based largely on Maxwell's complete circuit system, it is claimed to be infinity, although for a circuit as large as the circumference of the earth, a centimetre of which is not far from being absolutely straight, the energy calculated in this way is still quite small and by no means even approaches infinity. The writer's method

* Communicated by the Author.

[1] The paragraph forming the upper half of p. 243 contains an arithmetical error and should be deleted.

is so simple and direct that there should be no difficulty in finding an error in it if there were one; many have tried to find one, but none has been found to the writer's knowledge. The fact that the results do not agree with those of our forefathers, is, of course, no proper proof of error of the former, though many have claimed that it is. The explanation of the disagreement must therefore be looked for elsewhere, and the writer claims to have found that it lies in an error made in our older methods in the calculation of the energy residing in magnetic flux, under certain conditions; when this error is corrected, the results by both methods agree.

The purpose of the present article is to show that in calculating the stored energy in magnetic flux more care should have been taken not to add together again, as by integration, energies whose components had already been added. If in any system of component forces the total energy has been calculated from the resultant of these forces, it is evidently not proper to add to this the energies represented by its components; a resultant force is already a sum of its components. Yet this has been done in the past in calculating the energy stored in the flux, in certain cases.

As an analogy, suppose a vertical column of our atmosphere, disconnected from the rest, were allowed to drop a certain small distance; energy would thereby be set free. This can be correctly computed from the atmospheric pressure, 760 mm., the cross-section, and the distances through which it had moved. This is evidently the *total* energy of this movement, as this pressure is the *resultant* of the pressures of all the layers above the earth, to infinity, theoretically. It would evidently not be correct to then add to this the energy developed by the pressure at each layer above the bottom, moving through this same distance; this, if done by integration, would give an infinite amount of energy, which is known to be absurd. No more energy is set free by this motion than would be required to raise such a column of air through the same distance, and this is being done daily and is being correctly calculated.

Yet this error has been made in the analogous case of calculating the energy in magnetic flux under certain conditions. The radial pressure of the flux, encircling a single straight wire, is analogous to that of our atmosphere; that at the surface of the wire is the resultant of all the pressures radially beyond; each circular flux line in the theoretically infinite space around the

wire has a tension, according to Maxwell, which tension is manifested mechanically when a magnet attracts iron, hence is a true, mechanical force. Around a wire these tensions exert radial pressures, as a rubber band does when stretched around a cylinder. These pressures are superimposed on each other radially, each acting on the others between it and the surface of the wire, like the layers of the atmosphere. Hence that at the surface of the wire is the *resultant* of all those beyond, even extending to infinity, in the same sense as the 760-mm. pressure is the resultant of that of our atmosphere extending to infinity, theoretically.

These radial flux pressures in dynes per sq. cm., for any point in space, are calculated correctly by Maxwell's formula $H^2/8\pi$ in which H is the flux density, which at various distances r from the centre is given correctly by Maxwell's formula $H = 2i/r$ in which i is the current in C.G.S. units (10 amperes). The resultant pressure at the surface of the wire from these undisputed formulas is $p = i^2/2\pi R^2$, in which R is the radius of the wire, and p is in dynes per sq. cm. The total stored energy of this flux should therefore be calculated from this resultant pressure at the surface, as in the case of the atmosphere. To then calculate the energy in each differential layer beyond the surface, from this pressure formula $H^2/8\pi$ in that layer, and add them to the energy of this resultant pressure, by integration, as has been done in the past, would be as erroneous as it would be in the analogous and more evident case of our atmosphere, cited above.

There is, however, an important difference between this mechanical analogy and the flux, hence the analogy must not be carried too far. In the case of the atmosphere the quantity and weight which produce the pressures are the same after this energy has been set free as before; if it were dropped a second time through the same distance it would set free an equal amount of energy. Flux, however, is not a material, it is more like a stress or strain. If, therefore, a flux representing a given amount of energy sets free some of it, by relieving it of some of its strains, there is less of it left, and this reduction will appear throughout its whole field, even far distant from the surface of the wire. In discussing such fundamental actions one should, of course, as the writer has often pointed out, assume zero resistance, as is realizable at absolute zero of temperature. A current once started will then continue without a source until the magnetic energy in its

field is transformed into other energy; the square of the current will then decrease in proportion, as the stored energy decreases. The conditions are then not obscured by the addition of new energy from a source, as is always the case with the "constant current" usually specified in our older methods of studying such problems.

The stresses or strains forming the energy stored by this encircling flux outside of the wire, are more nearly analogous to those in a thick-walled, soft rubber tube containing a compressed liquid or gas. If the wall is radially infinitely thick and the hole in it is at first infinitely small and is then enlarged by forcing a liquid into it, and if its elasticity were like that of the ether, the analogy would be more complete, though it does not apply to the strains in the liquid which are quite different from those in a conductor. The radial inward pressure on the surface of the cylinder of water is then the resultant of all those beyond; the total energy in the whole wall was stored by the water pressure on this inside surface and it can be set free again there. When some of it is set free, there will be less left in the wall, even at great distances. To calculate the stored energy in this mechanical case in the way it has been done by others in the analogous magnetic flux case, would likewise give infinite energy, while it cannot, of course, be greater than the finite energy spent in producing these stresses and strains.

Starting either with this resultant radial pressure of the magnetic flux at the surface of the wire, or with the tangential tension corresponding to this radial pressure, and calculating from them the total stored energy per unit length, gives the same result, $i^2/2$ ergs per cm., as the writer had obtained by a different method, thereby confirming each other. This calculation, from the pressures, is given below.

The mechanical and the flux pressures do not correspond in different parts of the cross-section of the wire, though as the latter, together with the current, produce the former, their energies must, of course, be the same and that of the one may be calculated from that of the other. At the surface the flux pressures are greatest; this pressure from the outside is normally exactly in equilibrium with an equal but opposite flux pressure from the inside; when the mechanical pressure on the inside is relieved, that on the outside will act. At the centre the flux pres-

sure is zero and the mechanical pressure is a maximum. This and many other details are described in another paper by the writer.

The correction above described also affects some of our former calculations of flux and flux energy in the case of a coil. Assuming the simplest case of a very large single-turn circular circuit, each unit length is encircled by flux in very nearly the same way as a unit length of a straight wire; hence the same correction applies to both as the pressures at the surface of the wire are the resultant pressures of those radially beyond, and their only abutments are in the wire, hence the energies, calculated from all the various flux densities, cannot correctly be added together for the total.

It seems, however, that this error becomes less as the coil becomes smaller and as it approaches a solenoid, which may explain why it was not noticed before in coils of ordinary size used in practice. It may perhaps disappear almost entirely in the interior circular space of a toroid (a solenoid with a circular axis) in which the field is said to be of uniform density; when this density is quite uniform the pressures of all the lines on each other are equal, and are not resultants of those of the others.

The correctness of the formula for determining the energy from the number of lines (maxwells) and their magnetomotive force, namely, ergs = maxwells $\times$ $it/2$ in which i is the current in C.G.S. units (10 amperes) and t the number of turns, is not questioned, but it seems that it can be applied correctly only to a uniform field density in which all the lines are straight and parallel, all exerting the *same* repulsive pressures on each other in all directions, and none of them exerting a resulting pressure of any others; that is, when their tension does not exert a greater side pressure on one side than on another. But when they are curved and without abutments, like those encircling a wire, the inner ones represent the resultants of all the pressures beyond, as in the case of the circular layers of our atmosphere, and, in the opinion of the writer, the above formula cannot then apply correctly. All such forces from strains or stresses in the ether can have their abutments only in the material of the conductors or magnets, as the ether cannot, of course, form any abutment for them; hence the radial pressures of the flux around a wire must all appear in the wire, just as the straight tensile strains of the flux must all appear at the two unlike poles of the magnets. The

tensile strains in the case of the flux around the wire, are all transformed into radial pressures, as in the case of a stretched rubber band around a cylinder.

It can readily be shown that the e.m.f. induced by cutting or linking lines at the same rate, is quite independent of the flux energy residing in those lines, being dependent only on their number. Hence conversely nothing can be deduced concerning the energy residing in the flux from the e.m.f. induced.

When a flux line is cut or linked at a given rate, a definite e.m.f. is induced; when this is a counter e.m.f., as it is when a current is being started, its product with the current flowing at that moment, if completely in phase, represents the energy which is being stored at that moment by cutting that line. This energy is therefore a function of the current which happens to be flowing at that moment, and not of the energy in that line; and if it and the momentary e.m.f. are not in complete phase at that moment, their product is not the true energy. When the final steady current has been reached, the energy residing in this line has been claimed to be proportional to its m.m.f., that is, to the ampere-turns, hence to the current for a fixed number of turns. But it was shown above that this formula, ergs = maxwells $\times$ $it/2$, cannot be correctly applied when the pressures of some of the lines are resultants of the pressure of others. The growth of a starting current would then also, according to that formula, increase the energy in the lines *after* they had been cut when the current was smaller.

When flux from an external field is forced to be cut or linked with a closed circuit, as in every generator, the energy in the current comes from the driving power; but in the linking of self-generated flux, as in the starting of a current, there is no external mechanical driving force, and the lines may then be said to take their own time in being cut or linked; with coils having very high self-inductances this speed of cutting, which governs the counter e.m.f. induced, is known to be relatively quite slow compared with the case in which the number of lines is the same in a similar coil of lower self-inductance.

Moreover, many of the self-generated lines in a circuit in which a current is being started are cut by, or linked with, only a fractional part of the conductor, hence take only a fractional part in the induction, and have only a fractional part of the

m.m.f. As a hollow conductor has no flux on the inside, it may perhaps not yet be definitely known whether the flux lines from the filamentary conductors on the surface really cut the conductor, though it is true that they are then linked with it.

These and perhaps other considerations seem to indicate that there is still some looseness in our older system in the alleged definite relationship which seems to have been assumed in the past, apparently without adequate proof, between the flux lines cut or linked in their self-generation when a current is started, and the final stored energy in this flux. The quantity called self-inductance seems to be used indiscriminately for determining both, and assumes this relationship to be absolute, although all self-inductance formulas are only approximate. It seems to be still merely a postulate (an unproved assumption) instead of a well-established law. A revision seems to be needed, now that this basic constant $i^2/2$ ergs per cm. seems to be well established; it leads to many interesting deductions, one of which is that in the C.G.S. system a unit of current generates the equivalent of a unit of energy flux (perhaps better called resultant or condensed flux) per unit of length, in a fundamental single conductor.

The recognition of the fact that the energy represented by components should not be added to that represented by their resultant, leads to what might be called wattless flux, referring to that flux whose energy had already been accounted for, or attributed to other flux in the same field. This in turn leads to a very desirable distinction between several meanings of that ambiguous term "self-inductance," analogous to the important distinction between a true resistance and an impedance or a reactance (first made by Doctor Kennelly over thirty years ago) in the sense of a separation into an energy factor and a non-energy factor and which led to the important wattless ampere. This and other allied subjects are more fully discussed in another paper by the writer.

It follows from the above that $li^2/2$ represents the stored energy in a current flowing along a distance l, just as $mv^2/2$ represents the similar energy stored in a moving body. In the C.G.S. system they are, moreover, quantitatively equal to each other, both being ½ erg, when all the quantities are unity. Hence $li^2/2 = mv^2/2$, which means that if all the stored energy residing in a current i flowing over a distance l (under fundamental con-

dition, of course) were converted into that of a moving mass, the product of that mass and the square of its velocity will be numerically equal to the product li^2. The two sides of this equation have the same physical "dimensions," namely, that of energy.

The following is another proof of the constant $i^2/2$ ergs per cm. and is far simpler and much less involved than the one given in the writer's former article; the present one is based entirely on two of Maxwell's well-known and undisputed expressions, and does not involve either Northrup's or Kelvin's laws, and only simple mathematics. Incidentally it shows that Maxwell's $H^2/8\pi$ pressures in dynes per sq. cm. are real, mechanical pressures, when properly interpreted and applied; it seems that there had been some doubt about this. The following proof of the constant $i^2/2$ is based on the radial pressures, the proof based on the resultant tangential tensions of the encircling flux (deduced from the radial pressures) is quite as simple, if not more so, and leads to the same result.

An experimental proof that the energy stored in a body moving at a constant velocity is equal to that given by the well-known formula $mv^2/2$, might be obtained by opposing the motion of that body by a *constant* pressure or force until it comes to rest, when its energy is exhausted; then the product of this force and the distance over which it was applied before the body came to rest would evidently be equal to this $mv^2/2$ energy, thereby proving that this expression gives the correct amount, or that this amount might be determined in that way if that formula were unknown. Quite a similar method can be used to determine the amount of the analogous energy stored by a known current in a single, straight conductor (far removed from all others) of any given length l, using nothing but Maxwell's well-known laws and not involving any infinities, self-inductance, induction by linkage, or any mere postulates.

Assume any given length l of such a straight, single, liquid conductor of circular section having a radius R and a current I (C.G.S. units); or it may be the whole circuit of length l, if only it is large enough that each unit length is far enough from the return circuit not to be affected by it. Let the resistance of the circuit be assumed to be zero, hence a current once started will continue to flow without connection to any source, until all the energy W originally stored in its field has been set free. It

is well known and could easily be shown, that such a liquid conductor will tend to shrink radially, due to what are popularly known as the radial pinch pressures. In the present case these pressures are produced by this stored magnetic energy and not by a current from an external source, hence they must come to an end.

From Maxwell's $H = 2I/R$ for such single conductors, and his $H^2/8\pi$ pressure formula, this radial magnetic pressure on the outside surface is easily shown to be $I^2/2\pi R^2$ in dynes per sq. cm.; this is the total, *resultant* pressure. Whatever may be the detailed explanation of the mechanism of this shrinkage, say through a radial distance d, it is true in any case that this pressure must have then acted through this distance d and has thereby done work. Let the outflow of this liquid due to this shrinkage be assumed to be opposed by a *constant* mechanical pressure on the liquid (analogously to the constant pressure referred to above opposing a moving body), as for instance by making the liquid which is ejected by this shrinkage raise a weight, as it does in hundreds of electrical furnaces in daily use.[2] This opposing constant outflow pressure is made equal to the above radial pressure.

In thus acting on the outside through the distance d this constant radial pressure has set free a known part of the energy originally stored in the flux; this is determined from the distance d and the known force, equal to this constant pressure multiplied by the mean area. Hence there is then left less flux energy and therefore, of course, also less current. Let this shrinkage at constant outside pressure against an equal, constant, outflow pressure continue until the conductor has shrunk to a line, that is, to zero.

[2] If necessary to picture the details (though they are immaterial and do not affect the theory involved), let the liquid be supposed to be ejected through a tube leading to a cylinder foreign to the conductor, having a piston which raises the weight; as the pressures in the interior of the conductor are known to be different at different distances from the centre, this tube is assumed to be applied at that radial distance r from the centre (namely, when $r^2 = R^2/2$) at which this particular pressure exists and is constant during the shrinkage, hence it must be assumed to be moved toward the centre as the conductor shrinks. When this is done, and only under those conditions, the quantitative mathematical relations become extremely simple, as will be shown. This shrinkage against an opposing pressure must of course be assumed to take place simultaneously throughout the whole of that circuit, though only a portion of it needs to be considered mathematically.

At these radial and outflow pressures, assumed to be constant and equal, it can be shown that $I^2/i^2 = R^2/r^2$ in which i and r are the current and radius after a shrinkage; for the first pressure P is $I^2/2\pi R^2$ and the pressure p after this shrinkage to a radius r is $i^2/2\pi r^2$; when these are made equal to each other the above relation follows. Hence $I/i = R/r$, that is, the currents will diminish in proportion to the radius, and therefore the current, and with it of course the energy also, will become zero when the radius has shrunk to zero. The remaining energies are proportional to the squares of the currents or radii, but this is not an essential relation in this proof. This shows that *all* the stored energy has thus been consumed in crushing the conductor to zero. It also follows that for any given length l of the conductor the total radial *force*, as distinguished from pressure (force = pressure × area), diminishes in proportion to the radius, that is, $F/f = R/r$ and it is, therefore, also zero when the radius is zero. These simple relations hold only when the outflow pressure is made equal to this constant radial pressure; this outflow pressure always exists at a distance from the centre equal to the outside radius divided by the square root of 2, as shown by the Northrup formula and his experimental demonstrations show that these forces are true, mechanical forces, agreeing quantitatively with Maxwell's formulas.

As the original radial pressure is $P = I^2/2\pi R^2$, the force at first is $F = 2\pi RPl = lI^2/R$ in dynes, and as it diminishes in proportion to the radius, the work done by its acting radially to the centre is the mean of the original and zero, hence is $lI^2/2R$, which acting through the distance R gives as the original stored energy $W = lI^2/2$ ergs or $I^2/2$ ergs per cm. The same result could be obtained by the calculus.

Annual Fire Loss. (*Engineering World*, November, 1925.)—The property loss by fire for the past three years in round numbers amounts to more than $500,000,000 a year, a sum of money that would build two modern highways from Alaska to Cape Horn. It would build the Panama Canal and leave $150,000,000, sufficient to make the canal a sea-level waterway. This sum would pay the interest on the seven billion dollars Liberty Loan bonds for two years. Our annual fire loss is approximately $5 for every man, woman and child in the United States.

THE THEORY OF THE TELEPHONE RECEIVER.*

BY

W. H. INGRAM.

SUMMARY.

The equations connecting the winding current and diaphragm motion of a telephone receiver under impressed simple harmonic electromotive and mechanical forces are formulated under: (1) The Maxwell flux theory; (2) the magnetizing-winding theory; (3) the Ampèrean resistanceless fixed molecular circuit theory of permanent magnetism; and (4) a general molecular orbital theory of ferromagnetism. On the latter theory it is found that the coupling coefficients M and N in the equations in question are in general different, in counter-distinction to the other three theories as hitherto formulated. The flux theory is shown not to be inconsistent with this result when certain dynamical phase relations are introduced into the static flux-magnetomotive force formula (Ohm's Law analog) always employed in this theory. The Ampèrean rigid circuit theory is criticized as an inadequate dynamical representation of ferromagnetism. The magnetizing-winding theory (dynamical) is reproduced in slightly generalized form.

(1) THE FLUX THEORY.

Consider an electrodynamical mechanism (telephone receiver) consisting essentially of a winding, an iron core and an armature (diaphragm). The electromotive force equation for the winding may be written:

$$e = Ri + n\int \dot{B}\cdot \mathbf{n}\, d\sigma \tag{1}$$

where σ is to be taken over a surface bounded by the mean effective winding defined by the equation. The coefficient n is the number of effective turns. The equation may be also written:

$$e = Ri + n\frac{\partial \phi}{\partial t} + \xi nk\int d\sigma \frac{\partial}{\partial \xi} B\cdot \mathbf{n} \tag{2}$$

where k may be called a "surface factor," being equal to unity when all parts of the surface σ have the same velocity, ξ.

The mechanical effects are given by the Maxwell stresses at the surfaces of discontinuity in μ. The tractive force at such a surface is given by the vector equation:

$$\mathbf{F} = \frac{1}{8\pi}\int \mathbf{n}\frac{\mu_2 - \mu_1}{\mu_1}\left\{\frac{1}{\mu_2}(B_1\cdot \mathbf{n})^2 + \frac{1}{\mu_1}[B_1 \times \mathbf{n}]^2\right\} ds \tag{3}$$

where $\mathbf{n}$ is a unit vector normal to the surface and pointing into

* Communicated by the Author.

the medium 1 (the force on any element of area is thus always in the direction of the normal to said element); $(B\cdot\mathbf{n})^2 - B^2\cos^2(B, \mathbf{n})$, and $[B \times \mathbf{n}]^2 - B^2\sin^2(B, \mathbf{n})$. In the present case the medium denoted by the subscript *1* is air; by the subscript *2*, iron.

The formula (3) is derived from another given by Föppl-Abraham in vector form and originally by Maxwell in analytics. It should be known as the Maxwell pull-formula for electromagnets instead of the formula $F = B^2 s/8\pi$ which is current among engineers under that name. The latter is approximately valid, as an inspection of (3) makes clear, only when B is everywhere perpendicular to the surface s. A difference in the magnitudes of the coefficients of the dot and cross-products would seem to be a sufficient explanation of the wide variance of some experimental results from those given by the simpler formula.[1]

From the relation

$$\mathbf{F}\cdot\mathbf{n} = F = F_0 + \frac{\partial F}{\partial i}\delta i + \frac{\partial F}{\partial \xi}\delta\xi \tag{4}$$

and from (2), for the case of simple harmonic electromotive force e and mechanical force f, we have:

$$\left.\begin{aligned} e &= Zi + M\xi \\ f &= -Ni + z\xi \end{aligned}\right\} \tag{5}$$

where

$$F - F_0 = f,\ M = nk\int d\sigma \frac{\partial}{\partial x} B\cdot\mathbf{n},\ N = -\frac{\partial F}{\partial i} \text{ and } z = \frac{1}{j\omega}\frac{\partial F}{\partial \xi};$$

Z and z are the electrical and mechanical self-impedances and M and N are coupling coefficients. The resulting winding-current and velocity of the mechanical part will be simple harmonic when the coefficients in (5) are constants. This requires that $B = B_0 + \tilde{B}$, where the variable part $\tilde{B}$ is small in comparison with B_0.

[1] KALISCH, *Archiv für Elektrotechnik*, 1913, pp. 394, 459, 476. This author records experimental results five times greater than those computed by him on the simpler theory. The validity of the formula is questioned but no explanation offered. JEANS, "Electricity and Magnetism," 4th ed., p. 179, in treating of the hydrostatic pressures which occur when the constants of the medium vary, suggests that it is omitted terms of this kind which are responsible for the "lack of confirmation" of the Maxwell theory. Such effects very probably may be of importance in some cases but do not explain the discrepancies recorded by Kalisch and others and, moreover, as above pointed out, are not necessary. *Cf.* also GOUY, *Comptes Rendus*, Feb. 20, 1922; *Rev. Gén. d'El.*, Apr. 8, 1922.

The formulas (3) and (4) may be applied to a structure consisting of a long solenoid having a core divided by a centrally situated air-gap into a stationary part and a movable part connected with the diaphragm.[2] The surfaces σ and s are then equal, $k = 1$ and $B \times \mathbf{n} = 0$. Assuming $N = \Omega M$, where Ω may be a complex quantity, we have for this structure:

$$n\,\Omega \frac{\partial B}{\partial \xi} = \frac{\mu_2 - \mu_1}{4\pi\mu_1\mu_2} B \frac{\partial B}{\partial i} \tag{6}$$

the solution of which is, for large μ_2:

$$B = \frac{F_0 + \Omega 4\pi n i}{sR + (\xi_0 - \xi)} \tag{7}$$

where F_0 and $(sR + \xi_0)$ are constants of integration and have the usual significance.[3] This formula, with $\Omega = 1$, has been taken as the starting point in some treatments of the subject; the extension of its validity then beyond the static case is, of course, on pure assumption. It may be also written:

$$B = \Omega \frac{F_0' + 4\pi n i}{sR + (\xi_0 - \xi)} \tag{7'}$$

or

$$B = \Omega_1 \frac{F_0'' + \Omega_2 4\pi n i}{sR + (\xi_0 - \xi)} \tag{7''}$$

The possible importance of the Ω's (phase- or tensor-operators) for the inclusion of hysteresis effects would seem to be in line with the complex permeability theory which has been formulated for the same purpose.[4] The possibility of phase effects from other causes will be considered in §4.

(2) SIMPLE DYNAMICAL THEORY.

The system defined by the following functions will be discussed first:

$$\begin{aligned} T &= \frac{1}{2} l_0(\xi)\,\dot{y}^2 + \frac{1}{2} l_1(\xi)\,\dot{i}^2 + \frac{1}{2} \Sigma\Sigma l_{jk}(\xi)\,\dot{q}_j\dot{q}_k \\ &\quad + \nu(\xi)\,\dot{y}i + \Sigma\lambda_k(\xi)\,i\dot{q}_k + \Sigma\mu_k(\xi)\,\dot{y}\dot{q}_k + \frac{1}{2} m\dot{\xi}^2 \\ V &= \frac{1}{2} c(\xi_0 + \xi)^2 \\ F &= \frac{1}{2} r_0\dot{y}^2 + \frac{1}{2} r(\xi)\,\dot{i}^2 + \frac{1}{2} \Sigma\Sigma b_{jk}(\xi)\,\dot{q}_j\dot{q}_k + \frac{1}{2} \rho\dot{\xi}^2 \end{aligned} \tag{8}$$

[2] *Cf.* RAYLEIGH, *Phil. Mag.*, **38**, 1894, p. 295.

[3] A. E. KENNELLY and G. W. PIERCE, *Proc. Am. Acad. Arts and Sci.*, Sept., 1912.

[4] NUKIYAMA and SHOJI, "Tech. Reports of the Tohoku Imp. Univ.," Vol. 14, p. 21, 1924.

where $\dot{y}$ is the magnetizing current, i the winding current, $\dot{q}_k$ eddy currents and ξ the mechanical coördinate measured from the equilibrium position ξ_0.

The application of Lagrange's formula gives:

$$\left.\begin{aligned} E_0 &= r_0\dot{y} + \frac{d}{dt}\left\{l_0\dot{y} + \nu i + \Sigma_k \mu_k \dot{q}_k\right\} \\ e &= ri + \frac{d}{dt}\left\{\nu\dot{y} + l_1 i + \Sigma_k \lambda_k \dot{q}_k\right\} \\ 0 &= \Sigma_k b_{jk}\dot{q}_k + \frac{d}{dt}\left\{\mu_j\dot{y} + \lambda_j i + \Sigma_k l_{jk}\dot{q}_k\right\} \\ f &= -\frac{\partial T}{\partial \xi} + c(\xi_0 + \xi) + \rho\xi + m\ddot{\xi} \end{aligned}\right\} \tag{9}$$

Terms quadratic in the variables, occurring in $\frac{\partial T}{\partial \xi}$ and in the expansion: $\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\xi}\frac{\partial}{\partial \xi}$, when applied in the foregoing equations, may be made negligible on the assumption that $\dot{y} = \dot{y}_0 + \dot{\tilde{y}}$, $\dot{y} \gg i$, $\dot{y} \gg \dot{q}_k$ and $\dot{y}_0 \gg \dot{\tilde{y}}$, where $\dot{y}_0 = E_0/r_0$ is a large initial constant current on which an oscillatory part $\dot{\tilde{y}}$ is superimposed. We then obtain a system of linear equations on neglecting small quantities and from the sub-system represented by the third equation of (9), we obtain, by Cramer's rule:

$$\Delta\,\dot{q}_k = \sum_j \Delta_{jk}\alpha_j i + \sum_j \Delta_{jk}\beta_j \dot{\xi} \tag{10}$$

where

$$\left.\begin{aligned} \alpha_j &= j\omega[\lambda_j - j\omega\mu_j\nu/(r_0 + j\omega l_0)] \\ \beta_j &= [\mu_j' - j\omega\mu_j l_0'/(r_0 + j\omega l_0)]\dot{y}_0 \\ \gamma_{jk} &= b_{jk} + j\omega[l_{jk} - j\omega\mu_j\mu_k/(r_0 + j\omega l_0)] \\ \Delta &= -|\gamma_{ij}| \end{aligned}\right\} \tag{11}$$

Eliminating $\dot{\tilde{y}}$ and the $\dot{q}$'s from the second and fourth of (9) by the aid of the first and of (10) a simplified pair of equations results: $e = Zi + M\dot{\xi}$, $f = -Mi + z\ddot{\xi}$; where

$$M = [\nu' - j\omega\nu l_0'/(r_0 + j\omega l_0)]\,\dot{y}_0 + \frac{1}{2}\sum_j\sum_k \Delta_{jk}\alpha_j\beta_k/\Delta \tag{12}$$

$$Z = r + j\omega(l_1 - j\omega\nu^2/(r_0 + j\omega l_0)) + \frac{1}{2}\sum_j\sum_k \Delta_{jk}\alpha_j\alpha_k/\Delta \tag{13}$$

$$z = z_1 + l_0'^2\dot{y}_0^2/(r_0 + j\omega l_0) - \frac{1}{2}\sum_j\sum_k \Delta_{jk}\beta_j\beta_k/\Delta \tag{14}$$

and $z_1 = \rho + j\omega m + c/j\omega$ and the static equilibrium relation $c\xi_0 = -\frac{1}{2} l_0' y_0^2$ were used.[5]

A number of interesting conclusions may be drawn.[6] Thus the effect of the presence of eddies and of the magnetizing winding on the impedance to mechanical motion is proportional to y_0^2, or to the square of the polarizing flux.

In the absence of eddy currents the equation for M becomes: $M = \nu' - j\omega\nu l_0'/(r_0 + j\omega l_0)$. When the magnetizing winding coincides with the winding carrying the current i and when its resistance is negligible in comparison with ωl_0, *the equation for M becomes:* $M = \nu' - \nu l_0'/l_0 = 0$; for then $\nu = l_0$. When, however, the magnetizing winding includes a large external inductance in its circuit (making $l_0 \gg \nu$) the equation for M becomes: $M = \nu'$.

(3) THE AMPÈREAN CIRCUIT THEORY.

The only attempt at a dynamical theory of the permanent-magnet telephone receiver that has come to the writer's attention is that made by Poincaré,[7] who postulated a ferromagnetic mechanism consisting of a plurality of rigid Ampèrean circuits in which large initial currents circulate and which are presumably distributed throughout the body of the iron core. Now, the magnetizing winding of the previous paragraph becomes essentially a fixed large-scale Ampèrean circuit when $r_0 = 0$, $E_0 = 0$ and the circuit is closed internally, *i.e.*, has no external series inductance. We find then, *a posteriori*, that if this Ampèrean circuit (equivalent to a surface net of molecular circuits) coincides with the i-winding, that $M = N = 0$, or in other words, that no electromechanical interactions are possible! It is thus necessary in this theory to add specifications of a geometrical

[5] For a polarized receiver $\nu(\xi_0) = 0$, and when also the core is effectively laminated $\mu_k(\xi_0) = 0$; when the armature, $\lambda_k(\xi_0) = 0$. Note that l_{jk}, b_{jk}, λ_k and μ_k are functions of the frequency because of the skin effect. In the case of the induction-type receiver, where a closed coil is mounted on the diaphragm, q_1 (say) may be taken to be the current in the coil, $l_{11}(\xi)$, accordingly, its inductance, b_{11} its resistance, and $\lambda_1(\xi)$, $\mu_1(\xi)$ the mutual inductances between the coil and the i-winding and the magnetizing winding, respectively. The theory of this paragraph holds rigorously for an iron-free receiver of this type; for such a receiver l_0, l_1, l_{11} and ν are constant and $\Delta_{11}/\Delta = -z_0/(z_{11}z_0 + \omega^2\mu_1^2)$.

[6] *Cf.* R. L. WEGEL, *Jour. A. I. E. E.*, Oct., 1921.

[7] POINCARÉ, *L'Éclairage Électrique*, Feb. 16, 1907, p. 221.

nature to the premises in regard to the relation of the i-winding and the molecular circuits. This has seemed logically repugnant to the writer and has led him to question the adequacy of the theory.

Another obvious criticism of the Ampèrean circuit theory is that the flux through any such circuit remains constant, since $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_k}\right) = 0$, under variation of any of the coördinates, whereas in actual permanent magnets the flux increases as the reluctance of the path decreases, and *vice versa*.

(4) ORBITAL THEORY.

The electron theory of magnetism suggests that a theory of the forced vibrations about steady motion of a molecular system should have greater physical significance as a dynamical representation of ferromagnetism than the simpler Ampèrean molecular circuit theory.

Consider the system $L(Q_i, \dot{Q}_i, q_j, \dot{q}_j)$ treated by Whittaker,[8] in which Q_i are coördinates to be ignored and q_j are cyclic coördinates to be retained together with all non-cyclic coördinates. (The most obvious modification of the Ampèrean circuit theory to overcome the difficulties mentioned above would be the assignment of non-cyclic positional coördinates to the molecular circuits.) Thus two of the q_j are ξ and $\int i dt$. The coördinates Q_i and the remainder of q_j then specify the configuration of the molecular mechanism at any time.

The process of ignoration of the coördinates Q_i leads immediately to a Routhean function $R(q_k, \dot{q}_k)$; and if the system be assumed dissipative the force equations are given by

$$\phi_k = \frac{\partial F}{\partial \dot{q}_k} - \frac{\partial R}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_k}\right)$$

The oscillatory part of the ϕ's will be equal to e and f for the coördinates i and ξ, respectively, and equal to zero for the others. The inclusion of dissipation terms for other than these coördinates would approximate hysteresis, presumably. The equations for oscillatory motion in the coördinates q_k then are:

[8] "Dynamics," 2nd ed., § 83.

$$
\left.\begin{aligned}
e &= z_{11} i + (z_{12} + g_{12}) \dot{\tilde{q}}_2 + (z_{13} + g_{13}) \dot{\tilde{q}}_3 + \cdots + g_{1,n+1}\, \xi \\
0 &= (z_{21} + g_{21})\, i + z_{22} \dot{\tilde{q}}_2 + (z_{23} + g_{23}) \dot{\tilde{q}}_3 + \cdots + (z_{2,n+1} + g_{2,n+1})\, \xi \\
0 &= (z_{31} + g_{31})\, i + (z_{32} + g_{32}) \dot{\tilde{q}}_2 + z_{33} \dot{\tilde{q}}_3 + \cdots + (z_{3,n+1} + g_{3,n+1})\, \xi \\
&\vdots \\
f &= g_{n+1,1}\, i + (z_{n+1,2} + g_{n+1,2}) \dot{\tilde{q}}_2 + (z_{n+1,3} + g_{n+1,3}) \dot{\tilde{q}}_3 + \cdots \\
&\qquad + z_{n+1,n+1}\, \xi
\end{aligned}\right\} \quad (15)
$$

The impedances z_{ij} include the α's and β's of Whittaker's treatment and possibly dissipation coefficients and are symmetrical. The g_{ij}'s are "gyroscopic" and hence anti-symmetrical. Terms in $q_2 \ldots q_n$ constitute conditions for static equilibrium and cancel out. The algebraic elimination of the $n-1$ variables $\dot{\tilde{q}}_2, \dot{\tilde{q}}_3, \ldots \dot{\tilde{q}}_n$ through the $n-1$ equations whose left-hand sides are zero, as working out a simple case will show, *leads to equations of the form of* (5).

(5) THE ACTIVITY OF THE FORCES.

The activity of the forces is given by $\frac{1}{2}(e \cdot i + f \cdot \xi)$ and is equal to $Ri^2 + r\xi^2$ (root-mean-square values), where R and r are the scalar parts of Z and z, respectively,[9] of equations (5), provided M and N are scalar and equal, *or when one is the conjugate of the other*. In the dynamical theory of §2, where the molecular mechanism of the core was ignored, M and N were found to be equal scalars in the absence of eddies. The ferromagnetic mechanism of §3, specified by a single cyclic coördinate, was found to be inadequate to account for known physical facts. The foregoing considerations indicate that on a more complicated theory involving non-cyclic coördinates (such as orbital orientation coördinates) the conjugate relation between M and N, in the absence of eddies and hysteresis, would be the general and probable one. In the presence of the latter effects, M and N will be neither scalar equals nor tensor-operational conjugates and the activity equation will contain a cross-product term in ξ and i.

(6) A POSSIBLE APPLICATION; CONCLUSION.

The question of the equality of M and N becomes of practical importance in determining the optimum efficiency of a receiver (loud-speaker) for a given power source and load, or *vice versa*.

[9] *Phil. Mag.*, June, 1924.

Consider a generator whose internal e.m.f. is E and whose impedance is Z_2 and suppose that the mechanical (acoustic) load on the diaphragm is z_2. The force equations then are:

$$\left.\begin{aligned} E &= (Z_1 + Z_2)\, i + M\xi \\ 0 &= -Ni + (z_1 + z_2)\, \xi \end{aligned}\right\} \tag{16}$$

where Z_1 and z_1 are the electrical and mechanical impedances of the receiver. Whence: $\xi = EN/(Zz + MN)$, where $Z = Z_1 + Z_2$ and $z = z_1 + z_2$. It is customary in telephone practice to assume that E is a function of the characteristics of the generator of the form $|E| = k\sqrt{R_2}$. This assumption is principally justified by the fact that the alternating current generator equivalent to a carbon-button telephone transmitter in series with a source of constant e.m.f. has that voltage characteristic provided the ratio of the variable to the total resistance of the button is assumed constant for all transmitters of the same heat dissipating capacity.

The power expended in the load then is

$$P = |\xi|^2 r_2 = k\,(n^2 + \nu^2)\, r_2 R_2 / (a^2 + b^2) \tag{17}$$

where

$$\left.\begin{aligned} a &= (Rr - Xx + mn - \mu\nu) \\ b &= (Rx + rX + m\nu + n\mu) \\ M &= m + j\mu \\ N &= n + j\nu \end{aligned}\right\} \tag{18}$$

The conditions for a maximum power expenditure are

$$\left.\begin{aligned} \frac{\Delta}{r_2}\frac{\partial P}{\partial R_2} &= a^2 + b^2 - 2R_2(ar + bx) = 0 \\ \frac{\Delta}{2r_2R_2}\frac{\partial P}{\partial X_2} &= ax - br = 0 \\ \frac{\Delta}{R_2}\frac{\partial P}{\partial r_2} &= a^2 + b^2 - 2r_2(aR + bX) = 0 \\ \frac{\Delta}{2r_2R_2}\frac{\partial P}{\partial x_2} &= aX - bR = 0 \end{aligned}\right\} \tag{19}$$

where

$$\Delta = (a^2 + b^2)^2 / k\,(n^2 + \nu^2) \tag{20}$$

From the second and fourth equations of (19) we obtain:

$$x/r = X/R \tag{21}$$

and from the first and third:

$$r_1/R_1 = r_2/R_2 = r/R = x/X \tag{22}$$

Further algebraic manipulation gives:

$$\left.\begin{aligned} x_2 &= -x_1 - \frac{m\nu + n\mu}{2R_1} \\ X_2 &= -X_1 - \frac{m\nu + n\mu}{2r_1} \end{aligned}\right\} \quad (23)$$

and

$$\left.\begin{aligned} r_2^2 &= r_1^2 + (mn - \mu\nu)\frac{r_1}{R_1} - \frac{(m\nu + n\mu)^2}{4R_1^2} \\ R_2^2 &= R_1^2 + (mn - \mu\nu)\frac{R_1}{r_1} - \frac{(m\nu + n\mu)^2}{4r_1^2} \end{aligned}\right\} \quad (24)$$

These are the conditions for optimum power transference.

Measurements of M and N in verification of the present theory have not yet been made, but it is hoped that they can be reported upon in a later note. The main difficulties to be expected are the accurate determination of the mechanical impedance z and the velocity ξ, and the involved phase relations.

The Electrical Resistance of Mercury in Magnetic Fields. T. J. JONES. (*Phil. Mag.*, July, 1925.)—As long ago as 1891 Drude and Nernst noted a small increase in the resistance of a column of mercury when placed in a magnetic field transverse to its length. Other observers corroborated this result. The paper under review concerns itself with the change of resistance of a mercury column produced by applying a field perpendicular to the direction of the electric current in the liquid and further with the absolute change, and not with the change relative to the total resistance of the mercury. The selection of the absolute in preference to the relative change was determined by such an observation as the following. A column of the liquid 1 cm. long was put with the two current leads in the field, and the change in resistance amounted to 35×10^{-6} ohm. When a column eight times as long lay in the field, the leads, however, being outside of it, the change of resistance was less than one-third as much as before.

The total resistance of the mercury was of the order of .001 ohm. The range of field strength was from 2 to 10 kilogauss, of current strength from .5 to 5 amperes in a mercury column about 5 mm. in diameter. It was found that the absolute change of resistance was directly proportional to 1.58 power of the field strength and to the diameter of the mercury cylinder, and inversely proportional to the 0.4 power of the current strength and to the 0.41 power of the distance apart of the poles of the electromagnet furnishing the field. When this distance was varied, the intensity of the field along the axis was maintained constant. When the field was 10,000 gauss and the current 0.5 ampere, the change in resistance was .000248 ohm.

The change increased with the temperature of the mercury. Heating it from 17° to 290° increases the absolute change by about 12½ per cent.

The author holds that his results are consonant with a formula for the change in resistance derived by E. J. Williams in a paper in the same magazine. "The increase of resistance of a column of mercury (or any other liquid metal) when in a magnetic field is attributed to an expenditure of energy by the electric current in maintaining mass-motion of the mercury in certain portions of the column, this motion being developed by the electrodynamic action of the magnetic field on the mercury carrying the current." Under favorable conditions a lively motion of the mercury was actually observed. Furthermore, a few seconds were needed for the change of resistance to attain its full value, the mercury being in the magnetic field. This lag was not caused by any inductive effect in the apparatus. It seems to represent the time requisite for the mercury to attain its full motion. G. F. S.

Regarding Abruptly-beginning Magnetic Disturbances. L. A. BAUER and W. J. PETERS. (*Terrest. Magnetism and Atmos. Elec.*, June, 1925.)—This is not a new problem, for Doctor Bauer studied it in connection with the magnetic disturbances coincident with the eruption of Mount Pelée, May 8, 1902. A comparison of data from twenty-five magnetic observatories led him to conclude that the disturbance originated at about 75° west longitude and progressed eastward around the earth with a speed of about 7000 miles per minute, thus encircling the earth in three and one-half minutes. Later the same investigator examined the data for thirty-eight sudden disturbances in the interval 1882 to 1909, and found that they usually travel toward the east, though some move in the opposite direction, and that the speed is such that they make the circuit of the earth in three to seven minutes.

Fresh interest has been added to this investigation by one of the Agenda of the International Section of Terrestrial Magnetism and Atmospheric Electricity at Madrid in 1924. "Adoption of a scheme for observing at selected magnetic observatories the times of occurrence of 'sudden commencements' with special instruments admitting of a very high precision in the determination of time." In the present paper an examination is made of the records of six observatories on fifteen disturbances dating from 1906 to 1909, and of the data from five observatories for the same number of disturbances from 1922 to 1925. The former group occurred in a period of high sunspot activity, and the recent cases at a time of low sunspot activity. Both groups unite in indicating that the disturbance has its origin in low magnetic latitudes, and travels toward the magnetic poles. The speed of progression around the earth is 1000 km. or more, but about 1.8 minutes are required to go from the magnetic equator to the magnetic pole. G. F. S.

A BIOCHEMICAL STUDY OF BY-PRODUCT YEAST.*

BY

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THE poultry packer usually confines chickens in coops or "feeding batteries" and feeds them for a period of two weeks prior to slaughter. This feeding is known in the industry as "fleshing"; the object is to produce as great an increase as possible in their weight. The ration, as a rule, contains some by-product of the milling of wheat, such as bran, low-grade flour, red dog flour, middlings, as well as corn-meal and buttermilk. After the entry of the United States into the World War, the percentage of milled wheat converted into flour for human consumption was increased, and the yield of by-products of wheat was correspondingly decreased. As a result, the supply of these by-products available as feedstuffs was lessened; and substitutes had to be supplied. One of the substitutes used was distillers' grains, a by-product of the manufacture of industrial alcohol from corn.[1]

A ration containing 30 per cent. of whole corn ground to a fine meal, 10 per cent. of distillers' grains and 60 per cent. of buttermilk was more efficient than a ration containing only the corn-meal (40 per cent.) and buttermilk (60 per cent.). This greater efficiency was shown by the production of a greater gain in live weight in a given time with the consumption of less grain per pound of gain. Thus, when young chickens weighing, on the average, 1.4 pounds (635 grams) were fed the ration of corn-meal and buttermilk for a period of fourteen days, they gained 30.41 per cent. of their initial live weight, and consumed 3.55 pounds of grain per pound of gain. When young chickens of the same average weight were fed the ration of corn-meal, distillers' grains and buttermilk for fourteen days, they gained 35.01 per cent. of their initial live weight, and consumed 3.16 pounds of grain per pound of gain. The distillers' grains were derived from corn. In the process of fermentation, yeast was added to the corn, and carbohydrate was removed from it, while the relative amounts of protein and fat increased. Therefore, the greater efficiency of

* Communicated by the Author.

the distillers' grains was to be attributed, in part at least, to the yeast.

Both wet yeast and dried yeast—also by-products of the manufacture of industrial alcohol—were fed to young chickens. In the experiments with dried by-product yeast,[2] the average initial live weight of the chickens was 1.9 pounds (862 grams), the period of feeding fourteen days. The ration contained 36 per cent. of corn-meal, 4 per cent. of dried yeast, and 60 per cent. of buttermilk. The gain in live weight was 1.08 times that produced in control experiments by the ration of corn-meal and buttermilk previously described. For each pound of gain, 3.24 pounds of grain were consumed.

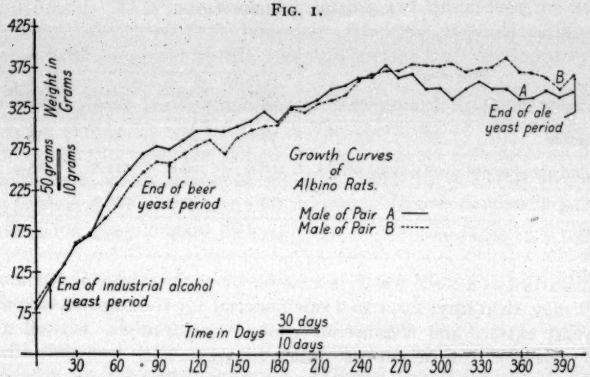
The results of the experiments on chickens prompted a study of by-product yeasts with respect to the presence in them of water-soluble B, the vitamin which promotes growth and protects from polyneuritis and beriberi. The by-product yeast used in the study included bottom yeast from an industrial alcohol plant, bottom yeast from a beer brewery, and yeast from an ale brewery. All three specimens had been dried by the Wittemann process.[3] In the industrial alcohol plant, a grade of molasses, which is a by-product of the sugar industry and is unfit for human consumption, is subjected to alcoholic fermentation by bottom yeast. In this process, from 1000 to 2000 pounds of yeast are produced daily. Analysis of the dried product showed it to have the following composition: Moisture, 8.73 per cent.; total solids, 91.27 per cent.; crude fat, 0.80 per cent.; crude protein, 31.57 per cent.; ash, 12.69 per cent.; crude fibre, 3.82 per cent.; nitrogen-free extract, 42.39 per cent. This dried yeast was reddish-brown in color. The other dried yeasts had a light yellow color. All three specimens had a flavor resembling that of meat extract.

The specimens of yeast were used successively as a constituent of a synthetic ration which was fed to two pairs of albino rats of the Tyler strain. The constituents of the ration and their proportion by weight were: Casein free from fat-soluble A and water-soluble B, 18; sucrose, 15; starch, 29.5; powdered agar, 5; salt mixture, 2.5; hydrogenated vegetable oil, 20; butter fat, 10; cod-liver oil, 4.3; dried yeast, 10. The salt mixture contained the following salts and supplied all necessary mineral food: Calcium lactate, monocalcium phosphate, magnesium sulphate, monosodium phosphate, sodium chloride, dipotassium phosphate, and

ferric citrate. The butter fat and cod-liver oil were added in order to insure the presence of the fat-soluble vitamins, A and D, in the ration; the former vitamin protects from xerophthalmia or keratomalacia; the latter vitamin plays a part in the prevention of rickets. Control experiments with albino rats on the ration minus the butter fat, cod-liver oil and yeast showed the absence of vitamins from its other constituents. Production of growth and preservation of health by the full ration, therefore, would show the presence of the water-soluble B vitamin in the yeast.

The dried yeast from the industrial alcohol plant was the

FIG. 1.



Growth curves of male rats.

same dried yeast as was used in the experiment with chickens. It was included in the ration of the rats for a period of eleven days. The dried beer yeast was then fed for a period of eighty-nine days, followed by the dried ale yeast for a period of 300 days. The study, therefore, extended over a total period of 400 days.

The rats grew, remained healthy, and reproduced. Their changes in body-weight are summarized in Table I. However, the maximum weights were attained by the male rats as early as the 120th and 150th day of the feeding of ale yeast, or the 220th and 250th day, respectively, of the entire study; thereafter their weight merely fluctuated. The growth curves of the male rats are given in Fig. 1. The curves were drawn from the author's data by his colleague, Prof. William G. Schmidt.

During the course of the study, the female of Pair *A* gave birth to three litters, a total of sixteen rats, of which fifteen survived at the end of the experiment. The female of Pair *B* gave birth to three litters, a total of ten rats, of which seven survived at the end of the experiment. The progeny were fed the same diet as the parents. One daughter of each pair likewise gave birth to a litter of five rats; the entire ten rats of the third generation were alive at the end of the experiment.

These results show that by-product yeast, which is frequently

TABLE I.

Body-weight of Albino Rats.

Body-weight.	Pair *A*.		Pair *B*.	
	Male. (Grams.)	Female. (Grams.)	Male. (Grams.)	Female. (Grams.)
Initial	77.6	76.6	88.7	55.1
End of industrial alcohol yeast period	107.5	99.8	112.8	65.4
End of beer yeast period	275.2	190.0	258.3	147.1
End of ale yeast period	345.6	261.5	365.7	245.6

discarded as a trade waste, is a source of water-soluble B vitamin. It may, therefore, serve as a raw material for the manufacture of yeast extract and commercial vitamin preparations, as well as a desirable constituent of foods for man and feedstuffs for domestic animals.

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[1] United States Department of Agriculture Bulletin No. 657, 12 pages, 1918.
[2] "Wheatless Rations to Flesh Young Chickens in Your Feeding Station," U. S. Department of Agriculture, Bureau of Chemistry, 10 pages.
[3] U. S. Patent 1,408,483.

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A NOTE ON THE CATHODE-RAY OSCILLOGRAPH.*

BY

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IT HAS been found that the use of a fluorescent screen deposited on a quartz window might greatly simplify the operation of the cathode-ray oscillograph and so extend its applications.

There exist at the present, two distinct types of this valuable instrument. That most commonly used is the Braun tube, or its modification due to J. B. Johnson.[1] The other type of oscillograph was suggested by Sir Joseph Thomson and developed by D. A. Keys[2] and A. B. Wood.[3]

Though the oscillograph has already proved its great value, certain difficulties yet remain in the operation of either type. In the Braun type, the cathode beam traces its deflection on a fluorescent screen inside the glass tube and the luminous streak is photographed with an ordinary camera. In order sufficiently to affect the photographic plate, the curve must be retraced many times by the cathode beam, and, as a result, only those phenomena which repeat themselves in identical cycles can be investigated with the Braun type.

In the Thomson type, the photographic plate is enclosed within the evacuated apparatus, so that the cathode beam impinges directly on the sensitized plate. Thus a single transit of the cathode beam will leave a distinct trace on the plate, and the instrument may be used for aperiodic or impulsive phenomenon, taking place in a very short time. But whenever a plate is to be inserted, or removed, the apparatus must be unsealed and then evacuated anew. This requires extra time, further technique and a permanently available pump.

It was thought that some of these disadvantages would be obviated by the use of a quartz window in the following manner. The window would consist of a thin, plane sheet of quartz sealed across the end of the oscillograph tube, opposite to the cathode.

* Communicated by Dr. A. S. Eve, Director, Department of Physics, McGill University and Associate Editor of this JOURNAL.

[1] JOHNSON: *J. Opt. Soc. Amer.*, **6**, p. 701, 1922.

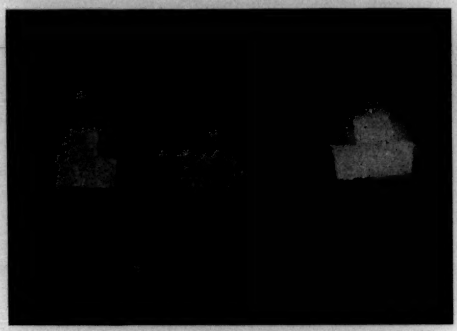
[2] KEYS: *Phil. Mag.*, **42**, p. 473, 1921; JOUR. FRANK. INST., **196**, p. 577, 1923.

[3] WOOD: *Proc. Lond. Phys. Soc.*, **35**, p. 109, 1923.

A fluorescent screen would be deposited on the inner surface of the quartz, and the photographic plate placed flat against the outer surface. Thus when the cathode beam traced a luminous curve on the screen, the curve would be reproduced directly on the photographic plate, eliminating, in this way, losses due to dispersion and absorption. By using quartz instead of glass which absorbs short wave-lengths, a single transit of the fluorescent spot should affect the plate.

The following simple arrangement was used to find whether

FIG. 1.



a. *b*. *c*.

a.—Exposure, order of $\frac{1}{180}$ of a second.
b.—Exposure, $\frac{1}{3}$ of a second.
c.—Exposure, $\frac{1}{2}$ a second.

the fluorescence could produce an image on the plate, through the quartz, with a very short exposure.

A straight glass tube (2.5 cm. diameter, 20 cm. length) was provided with a slightly concave aluminum cathode at one end and a plane quartz window (2.5 mm. thickness) at the other. A thin layer of willemite, zinc ortho-silicate, was deposited on the inner surface of the quartz to act as fluorescent screen. The anode was annular and placed near the middle of the tube which was exhausted by means of a "Cenco Hyvac" pump and excited by an induction coil of the ignition type. The photographic plate (Seed's Panchromatic) was placed against the outer face of the quartz window. In order to distinguish accidental fogging from the effect of the fluorescence, a screen of metal foil, with a T-shaped aperture in it, was placed between the quartz and the

sensitized plate. As in the case of the Braun tube, the effect of the faint glow of the discharge was quite negligible.

Various exposures were tried of one second and less. It was found, however, that, with the shortest exposure possible, a clear image of the aperture was formed on the plate by the fluorescence. This shortest exposure was made by a single interruption of the primary circuit of the induction coil, which gave rise to a very brief flash on the fluorescent screen. The figure shows very clearly how pronounced was the effect of the shortest exposure on the plate.

In this experiment no attempt was made to focus or concentrate the cathode rays. Clearly the effect would be greatly enhanced by using a Wehnelt cathode and a concentrated beam. Moreover, a mixture of calcium tungstate and zinc silicate, as described by J. B. Johnson,[4] would doubtless provide a fluorescent screen superior to the willemite screen used. The quartz sheet could be made thinner than the one actually employed, which had several flaws.

The shortest exposure made in this experiment could not exceed 1/100 of a second. It is estimated, however, that the average exposure of the plate to the moving spot is not less than 1/10,000 of a second. This value is obtained approximately from the expression, exposure $= d/v$ when d is the diameter and v the maximum velocity of the moving fluorescent spot. From the brightness of the image obtained, under the conditions of the experiment, it seems fairly certain that an exposure of one hundred times less duration would still leave a trace on the photographic plate. Also the persistence of the fluorescence on the screen would lengthen the exposure. It is obvious that this method can be applied as well to cases where the curves obtained are periodic.

SUMMARY.

Experiments are described which indicate that the cathode-ray oscillograph would be improved by the use of:

(*a*) A thin, transparent quartz window.

(*b*) A fluorescent screen of the greatest photographic effectiveness.

(*c*) A highly sensitive plate mounted *externally* in a plate holder forming part of the apparatus.

This system should have the following advantages:

[4] JOHNSON: *Loc. cit.*

(1) The tube being permanently sealed, the vacuum pump may be dispensed with.

(2) The plate being placed against the quartz, no system of lenses will be needed.

(3) The instrument will be suitable for the investigation of impulsive phenomena.

The writer is much indebted to Dr. D. A. Keys, of this department, for his constant interest and helpful advice.

The Spark Spectrum of Certain Metals in the Schumann Region. LEON and EUGENE BLOCH. (*Jour. Phys. et le Radium*, May, 1925.)—The correlation between the frequency of sunspots and the price of grain, phenomena of the earth's magnetism and other terrestrial variables has been examined with more or less fruitful results. It may not be amiss to examine the connection between the solar variable and the ups and downs in the number of papers published in physical magazines upon selected subjects. For instance, there was a time about a quarter of a century ago when N-rays were much discussed. Now they are merely of antiquarian interest, unless they, along with Sir Oliver Lodge's opinions on hot ice, warn that it is well to keep in mind and apply the excellent biblical advice that Prof. John Cox had carved on the entrance to the physical laboratory of McGill University: "Prove all things." Electrical waves and radioactivity in turn dominated the interest of investigators, and furnished vast numbers of papers. Now it is the spectrum that is to the fore, with its range from electrical waves miles in length down to the tiniest of gamma-rays. Great impetus has been given to this field by the conception of the method of emitting radiation that goes with Bohr's atom, by the development within a decade of measurements of the length of X-rays and of shorter waves and by the examination of the ultra-violet field carried out by Schumann and Lyman.

In the present paper the intensities and wave-lengths of 178 lines in the ultra-violet spectrum of manganese are tabulated. The range of lengths in Ångström units is from 1868.7 to 1464.5. Kayser, in his "Handbuch der Spektroskopie," gives no data for any line of this element shorter than 2215 Å. Indeed the authors know no previous work in this interval except Lane's list of twenty-six lines, published in 1924, and of his results they hold a poor opinion.

Chromium furnishes likewise about 180 lines, and copper approximately the same number. Silver, gold, and platinum present larger numbers of lines; silver about 300, gold about 389, and finally platinum 461, of which 279 are new. It is considered probable that the specimen of platinum used may have contained traces of other metals of the same family. In that case some of the lines would be due to these impurities. G. F. S.

AN ANALOGY BETWEEN PURE MATHEMATICS AND THE OPERATIONAL MATHEMATICS OF HEAVISIDE BY MEANS OF THE THEORY OF H-FUNCTIONS.*†

BY

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LINEAR FLOW OF HEAT IN A ROD OF LENGTH 1 (WITH RADIATION AT ENDS), DUE TO CONSTANT TEMPERATURE APPLIED AT t = 0.

This may be regarded as a problem in the flow of heat, or of electricity in a cable containing only resistance and capacity. The equations are

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\kappa}\frac{\partial v}{\partial t} \tag{207}$$

$$-\frac{\partial v}{\partial x} + hv = 0 \quad \text{when } x = 0 \tag{208}$$

$$+\frac{\partial v}{\partial x} + hv = 0 \quad \text{when } x = l \tag{209}$$

Two other conditions are required to complete the statement of the problem. For the first problem we will consider the cable cut at $x = y$, and a battery of zero resistance and constant E.M.F. E inserted at the moment $t = 0$. This would correspond to the rather fictitious heat problem of cutting the rod at $x = y$ and introducing a source of heat whose opposite faces were maintained at a constant difference of temperature E. However, the result will be used to derive the solution for the problem in heat where one end of the rod is suddenly raised to a constant temperature E and the other end is maintained at zero temperature.

Thus one condition is

$$(v_1)_{x=y} - (v_2)_{x=y} = E1 \tag{210}$$

where v_1 is the voltage on the left of the point $x = y$ and v_2 is the voltage on the right of $x = y$, and $E1$ denotes the H-function

$$\left.\begin{aligned} E1 &= 0 \quad & t &< 0 \\ E1 &= E \quad & t &> 0 \\ 0 \leq E1 &\leq E \quad & t &= 0 \end{aligned}\right\} \tag{211}$$

* Communicated by the Author.

† Concluded from p. 672 of the November issue.

The other condition will be taken to be

$$\left(\frac{\partial v_1}{\partial x}\right)_{x=y} - \left(\frac{\partial v_2}{\partial x}\right)_{x=y} = 0 \tag{212}$$

or $\frac{\partial v}{\partial x}$ is continuous at $x = y$, which corresponds to the assumption that there is no accumulation of heat or electricity at the point $x = y$.

Integrating (207) with respect to x

$$v_1 = (v)_{x=0} + x\left(\frac{\partial v}{\partial x}\right)_{x=0} + \int_0^x \int_0^x \frac{1}{\kappa}\frac{\partial v}{\partial t}\, dxdx \tag{213}$$

On consulting a paper by the author in the JOURNAL OF THE FRANKLIN INSTITUTE, June, 1923, it will be found that this may be written

$$v_1 = \cosh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)v_{x=0} + \frac{1}{\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}}\sinh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\left(\frac{\partial v}{\partial x}\right)_{x=0} \tag{214}$$

and using (208) this becomes

$$v_1 = \left[\cosh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right) + \frac{h}{\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}}\sinh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\right](v)_{x=0} \quad (x < y) \tag{215}$$

Similarly integrating (207) from l to x and using (209)

$$v_2 = \left[\cosh\left((x-l)\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right) - \frac{h}{\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}}\right.$$
$$\left.\times \sinh\left((x-l)\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\right](v)_{x=l} \qquad (x > y) \tag{216}$$

Hence from (210), letting $\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}} = \alpha$ for brevity,

$$\left[\cosh y\alpha + \frac{h}{\alpha}\sinh y\alpha\right](v)_{x=0}$$
$$-\left[\cosh(y-l)\alpha - \frac{h}{\alpha}\sinh(y-l)\alpha\right](v)_{x=l} = El \tag{217}$$

and from (212)

$$[\alpha \sinh y\alpha + h\cosh y\alpha](v)_{x=0}$$
$$-[\alpha\sinh(y-l)\alpha - h\cosh(y-l)\alpha](v)_{x=l} = 0 \tag{218}$$

Solving (217) and (218) we obtain after simplification

$$(v)_{x=0} = -\frac{\alpha[\alpha\sinh(y-l)\alpha - h\cosh(y-l)\alpha]El}{(\alpha^2+h^2)\sinh l\alpha + 2h\alpha\cosh l\alpha} \tag{219}$$

Hence

$$v_1 = -\frac{[\alpha \cosh x\alpha + h \sinh x\alpha]\ \alpha \sinh (y-l)\alpha - h \cosh (y-l)\alpha]E1}{(\alpha^2+h^2)\sinh l\alpha + 2h\alpha \cosh l\alpha} \quad (x<y) \tag{220}$$

with a similar expression for v_2 when $x > y$.

Now if (220) is written in the form

$$\left[\left(\frac{1}{\kappa}\frac{\partial}{\partial t}+h^2\right)\frac{1}{\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}}\sinh\left(l\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)+2h\cosh\left(l\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\right]v_1$$

$$= -\left[\cosh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)+\frac{h}{\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}}\sinh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\right]\times$$

$$\left[\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\sinh\left((y-l)\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)-h\cosh\left((y-l)\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\right]E1 \tag{221}$$

and the expansions for $\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\sinh\left(l\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)$ and $\cosh\left(l\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)$, etc., are substituted, it may be regarded as a linear differential equation of infinite order for v_1. In order to solve it, the operator on v_1 must be written in the form of an infinite product. This can be done by finding its roots when it is regarded as a function of $\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}$. Thus the roots of the equation in α,

$$\tanh l\alpha = -\frac{2h\alpha}{\alpha^2+h^2} \tag{222}$$

must be found. The roots of this equation are all imaginary (see Carslaw, "Theory of Heat Conduction," 1921, p. 75 *et seq.*). Writing $\alpha - j\beta$ the equation becomes

$$\tan l\beta = \frac{2h\beta}{\beta^2-h^2} \tag{223}$$

The roots of this equation are all real, and the negative ones equal to the positive ones in absolute value. If $\beta_1, \beta_2, \ldots \beta_n \ldots$ are the roots of this equation, then (221) can be written in the form

$$\left(\frac{\frac{1}{\kappa}\frac{\partial}{\partial t}}{\beta_1^2}+1\right)\left(\frac{\frac{1}{\kappa}\frac{\partial}{\partial t}}{\beta_2^2}+1\right)\cdots\cdots\left(\frac{\frac{1}{\kappa}\frac{\partial}{\partial t}}{\beta_n^2}+1\right)\cdots\cdots v_1 \longleftrightarrow f\left(\frac{d}{dt}\right)E1 \tag{224}$$

where $f\left(\frac{d}{dt}\right)$ is an infinite series in ascending powers of d/dt. In order to solve this equation as shown in the paper in the

JOURNAL OF THE FRANKLIN INSTITUTE, June, 1923, it is necessary to find functions $A_1, A_2, \ldots A_n, \ldots$ such that

$$\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right) A_n = E1 \tag{225}$$

In the paper referred to, the solution of the equations obtained there similar to (225) was written in the form

$$A_n = E\left(1 - \gamma_n e^{-\kappa\beta^2 n t}\right) \tag{226}$$

where γ_n denoted an arbitrary constant. As previously shown in this paper, however, when $E1$ is the Heaviside unit function, there are no arbitrary constants involved and the solution of (66) is

$$A_n = E\left(1 - e^{-\kappa\beta^2 n t}\right) 1 \tag{227}$$

Then following the scheme outlined in the JOURNAL OF THE FRANKLIN INSTITUTE, June, 1923, the method of elimination has to be applied to the following infinite system of equations

$$\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right)\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right) \cdots\cdots\cdots \left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right) \cdots\cdots v$$

$$\longleftrightarrow f\left(\frac{\partial}{\partial t}\right) E1\left(1 - e^{-\kappa\beta^2_1 t}\right)$$

$$\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right)\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right) \cdots\cdots\cdots \left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right) \cdots\cdots v$$

$$\longleftrightarrow f\left(\frac{\partial}{\partial t}\right) E1\left(1 - e^{-\kappa\beta^2_2 t}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right)\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right) \cdots \left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right)\left(\frac{1}{\kappa}\frac{\partial}{\partial t} + 1\right) \cdots v$$

$$\longleftrightarrow f\left(\frac{\partial}{\partial t}\right) E1\left(1 - e^{-\kappa\beta^2_n t}\right) \tag{228}$$

When the number of roots in question is finite it was shown in the JOURNAL OF THE FRANKLIN INSTITUTE, June, 1923, that Heaviside's expansion theorem supplies the most expeditious means of solving such a system of equations. Let us assume that the expansion theorem holds when the number of roots becomes infinite, and apply it to (220) when $j\beta$ is substituted for α. Then

$$v_1 = -\frac{(\beta\cos\beta x + h\sin\beta x)\,[\beta\sin(y-l)\,\beta + h\cos(y-l)\beta]\,E1}{(\beta^2-h^2)\sin l\beta - 2h\beta\cos l\beta}$$

$$= \frac{Y(\beta)}{Z(\beta)}E1. \tag{229}$$

$$\frac{Y(o)}{Z(o)} \longleftrightarrow -\frac{(1+hx)h}{-h^2l-2h} \longleftrightarrow \frac{1+hx}{2+hl}. \tag{230}$$

$$\frac{dZ}{d\beta} = 2\beta\sin l\beta + (\beta^2-h^2)\,l\cos l\beta - 2h\cos l\beta + 2hl\beta\sin l\beta$$

$$= \cos l\beta\,[2\beta(1-hl)\tan l\beta + (\beta^2-h^2)\,l + 2h] \tag{231}$$

On substituting the root β_n in (231) it becomes after simplification

$$\left(\frac{dZ}{d\beta}\right)_{\beta_n} = \frac{[(\beta^2+h^2)\,l+2h]\,[\beta^2+h^2]}{\beta^2-h^2}\cos\beta_n l \tag{232}$$

Also substituting the root β_n in

$$\beta_n\sin\beta_n(y-l) + h\cos\beta_n(y-l)$$

$$= \sin\beta_n y\,[\beta_n\cos\beta_n l - h\sin\beta_n l] - \cos\beta_n y\,[\beta_n\sin\beta_n l + h\cos\beta_n l]$$

$$= \cos\beta_n l\sin\beta_n y\left[\beta_n + \frac{2h^2\beta_n}{\beta^2_n - h^2}\right] - \cos\beta_n l\cos\beta_n y\left[\frac{2h\beta^2_n}{\beta^2_n - h^2} - h\right]$$

$$= \cos\beta_n l\,[\beta_n\sin\beta_n y - h\cos\beta_n y]\,\frac{\beta^2_n + h^2}{\beta^2_n - h^2} \tag{233}$$

Hence since

$$v_1 = \frac{Y(o)}{Z(o)} + \Sigma\frac{Y(p_n)\,e^{-p_n t}}{p_n\left(\frac{dZ}{dp}\right)_{p=p_n}} \tag{234}$$

where $p = \frac{d}{dt}$

$$v_1 \longleftrightarrow E1\frac{1+hx}{2+hl}$$

$$-E1\sum_{n=1}^{n=\infty}\frac{(\beta_n\cos\beta_n x + h\sin\beta_n x)\,(\beta_n\sin\beta_n y - h\cos\beta_n y)}{\frac{1}{2}\beta\,[(\beta^2+h^2)\,l+2h]}\,e^{-\kappa\beta^2_n} \tag{235}$$

with a similar expression for v_2 when $x > y$.

As a special case consider $h=\infty$ and $y=l$. Then β_n is a root of

$$\tan \beta l = 0 \text{ or } \sin \beta l = 0 \tag{236}$$

the roots of which are

$$\beta = \frac{\pi}{l}, \frac{2\pi}{l}, \ldots \frac{n\pi}{l} \ldots \tag{237}$$

and v reduces to

$$v \longleftrightarrow \frac{Ex}{l} 1 + \sum_{1}^{\infty} \frac{(-1)^n El}{n} \sin \frac{n\pi x}{l} \; e^{-\kappa \frac{n^2\pi^2}{l^2} t} \tag{238}$$

the well-known result for a short-circuited cable or linear flow of heat in a rod whose temperature at one end is kept equal to zero, and the other end is maintained at a constant temperature E, which is applied at the moment $t=0$. Equation (235) is not a suitable form from which to derive the case of an open-circuited cable since then (208) would be written $\frac{\partial v}{\partial x}=0$ and (209) would be replaced by $v=El$ when $x=l$, but this case can be easily derived directly.

Let us consider the points of difference between this method and the standard method of procedure. It will be noticed that the equation

$$v = 0 \text{ when } t = 0$$

which is usually taken as one of the initial conditions, has not been explicitly stated. It is implicitly contained in (210) (the system being a dissipative one) since $El=0$ when $t<0$.

By use of the H-function El, it has not been necessary to use unknown constants in (227), which is the solution of (225).

Attention may again be called to the fact that in writing (221) in the expanded form (224), all that can be proved in the theory of H-functions, is that the difference between the two equations, regarded as an H-function of $\frac{1}{\kappa}\frac{\partial}{\partial t}$ operating on v, can be shown to be less than ε, and for this reason the sign $\longleftrightarrow$ is used instead of the sign $=$ in the subsequent equations.

It has further been assumed that the Heaviside expansion theorem gives the solution of equations (228). It can easily be proved by induction as suggested in the JOURNAL OF THE FRANKLIN INSTITUTE, June, 1923, p. 823, that if it holds for n factors it holds for $n+1$, and since it holds for $n=1$ it must hold

in general. But another difficulty arises here in connection with (228), for we have not proved, as we proved in the case where $f\left(\frac{d}{dt}\right)$ contains a finite number of terms, that terms of the type $\left(\frac{d}{dt}\right)^2 El$, etc., do or do not exist. A discussion of this point would be very difficult if not impossible since $f\left(\frac{d}{dt}\right)$ contains an infinite number of terms in this case. A method of avoiding this difficulty will be given later in this paper.

LINEAR FLOW OF HEAT IN A ROD OF LENGTH 1 DUE TO AN INSTANTANEOUS POINT SOURCE.

Consider the same rod or cable as is given by (207), (208) and (209), but assume that a quantity of heat or electricity is placed at the point $x = y$. This can be expressed by the equation

$$-\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} = RQ\frac{dl}{dt} \text{ when } x = y \tag{239}$$

since if this equation is integrated with respect to t

$$\int_{t_1}^{t_2}\left(-\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x}\right)_{x=y} dt = RQ|1|_{t_1}^{t_2} \tag{240}$$

If $t_1 < 0$ and $t_2 < 0$ the value of the expression on the right-hand side is zero, if $t_1 < 0$ and $t_2 > 0$ it is equal to RQ, and if $t_1 > 0$ and $t_2 > 0$ it is equal to zero. The left-hand side, however, is proportional to the current flowing at the points $x = y - 0$ and $x = y + 0$, and thus equation (239) says that these currents are equal, and that there is no accumulation of heat or electricity at the point $x = y$ except at the moment $t = 0$, and that the difference between the currents at that instant is such that the time integral from $t = 0 - \varepsilon$ to $t = 0 + \varepsilon$ is equal to the charge on what may be considered as an infinitesimal condenser at the point $x = y$. R is the resistance per unit length of cable in electrical problems, and may be taken equal to unity in heat problems.

Also

$$v_1 = v_2 \text{ when } x = y \tag{241}$$

where v_1 is the temperature when $x < y$, and v_2 when $x > y$. Equation (241) then says that there is no constant source of E.M.F. or heat inserted at $x = y$.

Proceeding as in the last example, we again obtain (215) and (216)

$$v_1 = \left[\cosh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right) + \frac{h}{\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}}\sinh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\right](v)_{x=0} \quad x<y \quad (242)$$

$$v_2 = \left[\cosh\left((x-l)\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right) - \frac{h}{\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}}\sinh\left((x-l)\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)\right](v)_{x=l} \; x>y \quad (243)$$

using (239) and (241) and writing $\alpha = \sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}$, we obtain

$$-[\alpha\sinh y\alpha + h\cosh y\alpha](v)_{x=0}$$

$$+[\alpha\sinh(y-l)\alpha - h\cosh(y-l)\alpha](v)_{x=l} = RQ\frac{dl}{dt} \quad (244)$$

and

$$\left[\cosh y\alpha + \frac{h}{\alpha}\sinh y\alpha\right](v)_{x=0}$$

$$-\left[\cosh(y-l)\alpha - \frac{h}{\alpha}\sinh(y-l)\alpha\right](v)_{x=l} = 0 \quad (245)$$

Solving these equations for $(v)_{x=0}$ and $(v')_{x=l}$ and substituting in (242) and (243), we obtain

$$v_1 = -\frac{\alpha\left[\cosh x\alpha + \frac{h}{\alpha}\sinh x\alpha\right]\left[\cosh(y-l)\alpha - \frac{h}{\alpha}\sinh(y-l)\alpha\right]}{(\alpha^2+h^2)\sinh l\alpha + 2h\alpha\cosh l\alpha}RQ\frac{dl}{dt} \quad (246)$$

with a similar equation for v_2. Applying the Heaviside expansion theorem, we get

$$v_1 \longleftrightarrow 2RQl\sum_{n=1}^{n=\infty}\frac{[\alpha_n\cos\alpha_n x + h\sin\alpha_n x]\,[\alpha_n\cos\alpha_n y + h\sin\alpha_n y]}{(\alpha_n^2-h^2)l+2h}e^{-\kappa\alpha^2_n t} \quad (247)$$

where the summation is over the roots of the denominator of (222).

GREEN'S FUNCTIONS.

Equations (235) and (247) with the corresponding equations for v_2 will be called the two Green's functions G_A and G_B for what may be called the "space" denoted by equations (207), (208) and (209). There is at present some confusion in the use of the term, Green's function. For instance, Carslaw in his "Mathematical Theory of Heat Conduction," Chap. X, defines the Green's function as the temperature at the time t due to an instantaneous point source of strength unity generated at the point $P(x', y', z')$ at the time t_0, the solid being initially at zero temperature, and the surface being kept at zero temperature.

But in Chap. XII he defines the Green's function for the steady temperature equation as a continuous function $K(x, \xi)$ which satisfies the equation for steady temperature, and the same boundary conditions, while its differential coefficient with respect to x, denoted by $K'(x, \xi)$ is continuous except at $x = \xi$, where it is discontinuous in such a way that $[K'(x,\xi)]_{\xi+0}^{\xi-0} = 1$. He also points out that these two definitions must not be confused.

The Green's function used by Carslaw in Chap. X is what we have called G_B with some modifications. We shall not have any occasion to use a Green's function corresponding to the one defined by Carslaw in Chap. XII. Hilbert in his work on integral equations has classified his Green's functions according to the form of the equations (208) and (209). For instance, he calls G_I the steady state solution of (207) when

$$\left.\begin{aligned} v &= 0 \text{ when } x = 0 \\ v &= 0 \text{ when } x = l \end{aligned}\right\} \tag{248}$$

and G_{II} the steady state solution when

$$\left.\begin{aligned} v &= 0 \text{ when } x = 0 \\ \frac{\partial v}{\partial x} &= 0 \text{ when } x = l \end{aligned}\right\} \tag{249}$$

and G_{III}, G_{IV}, etc., correspond to other modifications.

It has seemed best here to define G_A as the temperature at a time t due to a steady source at the point $P(x', y', z')$ whose temperature is zero before the time $t = 0$, and constant and equal to unity after the time $t = 0$, where the space or solid we are considering is given by equations (207), (208) and (209) or other similar equations. G_B is the temperature at the time t due to an instantaneous point source of strength unity generated at the point $P(x', y', z')$ at the time $t = 0$. The additional specification that the solid is at zero temperature when $t = 0$ must be used for a non-dissipative system; this is not necessary for dissipative systems since the statement that the source was of zero strength before the time $t = 0$ is sufficient.

For practical purposes it has been shown, in the one-dimensional case, that G_A is defined by

$$\left.\begin{aligned} \left[v\right]_{y+0}^{y-0} &= E1 \\ \text{and} \quad \left[\frac{\partial v}{\partial x}\right]_{y+0}^{y-0} &= 0 \end{aligned}\right\} \tag{250}$$

in addition to (207), (208) and (209) or similar equations.

G_B is defined by

$$\left.\begin{aligned}\left[v\right]_{y+o}^{y-o} &= o\\ \left[\frac{\partial v}{\partial x}\right]_{y+o}^{y-o} &= C\frac{dl}{dt}\end{aligned}\right\} \qquad (251)$$

in addition to (207), (208) and (209) or similar equations, and where C is some constant which can be determined from the particular problem under consideration. It has already been shown how the determination of the Green's function becomes an automatic process by the use of the Heaviside expansion theorem with these equations.

By making use of G_A and Duhamel's theorems (see Carslaw, "Theory of Heat," pp. 17 and 18) problems involving the introduction of a source of heat whose temperature varies with the time can be solved. For instance, we have already found G_A for a linear rod of length l, one of whose ends is kept at zero temperature, and the other end at a temperature unity when $t > o$, and zero when $t < o$. This is given in (238). This modified solution of the general equation (235) is taken, as the result to be presently obtained may be more easily checked, and it illustrates the method quite as well. From (238)

$$v \longleftrightarrow \frac{Ex}{l} + \frac{2E}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\,e^{-\kappa\frac{n^2\pi^2}{l^2}t} \qquad (252)$$

To find the temperature if the end $x = l$ is kept at a temperature $E\cos wt$ applied at the instant $t = o$, instead of the constant value E applied at the instant $t = o$ in (252), we have by Duhamel's theorem (*loc. cit.*)

$$v = \int_o^t \phi(\lambda)\frac{\partial}{\partial t}F(x,y,z,t-\lambda)\,d\lambda \qquad (253)$$

which in this case becomes

$$v \longleftrightarrow \int_o^t \cos\omega\lambda\,\frac{\partial}{\partial t}\left[\frac{Ex}{l} + \frac{2E}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\,e^{-\kappa\frac{n^2\pi^2}{l^2}(t-\lambda)}\right]d\lambda \qquad (254)$$

and on performing the differentiation and integration

$$v \longleftrightarrow \text{Permanent term} + \frac{2E}{l}\sum_{1}^{\infty}(-1)^n\frac{\frac{n^3\pi^3}{l^3}}{\frac{n^2\pi^4}{l^4}+\frac{\omega^2}{\kappa^2}}\sin\frac{n\pi x}{l}\,e^{-\kappa\frac{n^2\pi^2}{l^2}t} \qquad (255)$$

The permanent term may be derived by other well-known methods, although of course it may be done here by summing the series involved if so desired, but it is tedious to do so. The transient term may be checked with the result given by Carslaw, "Theory of Heat Conduction," p. 212.

Using G_B, problems involving an initial distribution of heat or electricity can be solved by integration. Take the rod given by (207), (208) and (209), and let the initial temperature be given by

$$v = f(x) \text{ when } t = 0 \tag{256}$$

This initial temperature $v = f(x)$ may be regarded as being made up of a number of instantaneous point sources of strengths proportional to the value of $f(x)$ at each point in the bar. Hence making $RQ = 1$, we get for the temperature due to an initial distribution $f(x)$

$$v = \int_0^l G_B(x,y) f(y)\, dy \tag{257}$$

or

$$v \longleftrightarrow 2 \sum_1^\infty e^{-\kappa\alpha^2_n t} \frac{[\alpha_n \cos \alpha_n x + h \sin \alpha_n x]}{(\alpha^2_n - h^2) l + 2h} \int_0^l f(x) (\alpha_n \cos \alpha_n x + h \sin \alpha_n x)\, dx \tag{258}$$

using equation (247) where $\alpha_1, \alpha_2, \ldots \alpha_n \ldots$ are the roots of

$$\tan \alpha l = \frac{2\alpha h}{\alpha^2 - h^2} \tag{258}$$

which is the solution given by Carslaw, "Theory of Heat," p. 77.

It should now be apparent that when the Green's functions G_A and G_B have been obtained for any particular space, the solution for any given impressed E.M.F. and any arbitrary initial condition can be obtained by integration, and thus the determination of these functions becomes of the greatest importance. It should also be apparent why we have chosen to classify the Green's functions as G_A and G_B, corresponding to the permanent and instantaneous sources rather than use the classifications used by Hilbert and others. It is evident that by using a combination of G_A and G_B we can obtain the solution for any arbitrary impressed E.M.F. and initial conditions. It may perhaps be remarked here the G_B is the really important function, as a method will be shown later by which G_A may be determined from G_B.

FLOW OF HEAT IN AN INFINITE HOLLOW CYLINDER, SURFACES $r=a$ AND $r=b$, KEPT AT ZERO TEMPERATURE.

This example is taken to show how the Green's function can be determined in a case in which the properties of the functions which arise are not as familiar as those of the trigonometric functions. Take an infinite hollow cylinder. Surfaces $r=a$ and $r=b$ kept at zero temperature, with an initial instantaneous source of heat at $r=r'$. The equations to be solved are

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}=\frac{1}{\kappa}\frac{\partial v}{\partial t} \tag{259}$$

$$v=0 \qquad r=a \tag{260}$$

$$v=0 \qquad r=b \tag{261}$$

$$-\frac{\partial v_1}{\partial r}+\frac{\partial v_2}{\partial r}=Q\frac{d1}{dt} \qquad r=r' \tag{262}$$

$$v_1=v_2 \qquad r=r' \tag{263}$$

where v_1 is the value of v when $r<r'$ and v_2 the value of v where $r>r'$.

The general solution of (259) is [5]

$$v=AJ_0(\alpha r)+BH_0(\alpha r) \tag{264}$$

where

$$\alpha=\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}} \tag{265}$$

Hence when $r<r'$ from (260)

$$v_1=[J_0(\alpha r)\,H_0(\alpha a)-J_0(\alpha a)\,H_0(\alpha r)]C_1 \tag{266}$$

and when $r>r'$ from (261)

$$v_2=[J_0(\alpha r)\,H_0(\alpha b)-J_0(\alpha b)\,H_0(\alpha r)]C_2 \tag{267}$$

where C_1 and C_2 are functions of t, but independent of r. When $r=r'$ we get from (262) and (263)

$$v_1\frac{\partial v_2}{\partial r}-v_2\frac{\partial v_1}{\partial r}=Qv_2\frac{d1}{dt} \qquad r=r' \tag{268}$$

or

$$\begin{aligned}&\{J_0(\alpha r')H_0(\alpha a)-J_0(\alpha a)H_0(\alpha r')\}\{J_0'(\alpha r')H_0(\alpha b)-J_0(\alpha b)H_0'(\alpha r')\}\alpha C_1\\ &-\{J_0(\alpha r')H_0(\alpha b)-J_0(\alpha b)H_0(\alpha r')\}\{J_0'(\alpha r')H_0(\alpha a)-J_0(\alpha a)H_0'(\alpha r')\}\alpha C_1\\ &=[J_0(\alpha r')H_0(\alpha b)-J_0(\alpha b)H_0(\alpha r')]\,Q\frac{d1}{dt}\\ &=\kappa_1\alpha C_1\end{aligned} \tag{269}$$

To simplify κ_1, we have that if v_1 and v_2 are any two independent solutions of Bessel's equations (see Gray and Matthews, "Bessel Functions," p. 24)

[5] $H_0(\alpha r)$ is generally written $H_0^{(1)}(\alpha r)$.

$$v_1(\alpha r') \frac{\partial}{\partial(\alpha r')} v_2(\alpha r') - v_2(\alpha r') \frac{\partial}{\partial(\alpha r')} v_1(\alpha r') = \frac{A}{\alpha r'} \tag{270}$$

To determine the constant A in this case, let $r' = b$. Then

$$[J_0(\alpha b)H_0(\alpha a) - J_0(\alpha a)H_0(\alpha b)]\,[J_0'(\alpha b)H_0(\alpha b) - J_0(\alpha b)H_0'(\alpha b)] = \frac{A}{\alpha b} \tag{271}$$

Hence

$$\kappa_1 = \frac{\alpha b}{\alpha r'}\,[J_0(\alpha b)H_0(\alpha a) - J_0(\alpha a)H_0(\alpha b)]\,[J_0'(\alpha b)H_0(\alpha b) - J_0(\alpha b)H_0'(\alpha b)] \tag{272}$$

This last expression can again be simplified since

$$\alpha H_0(\alpha b) \frac{d}{d(\alpha b)} J_0(\alpha b) - \alpha J_0(\alpha b) \frac{d}{d(\alpha b)} H_0(\alpha b) = -\frac{2i}{\alpha b \pi} \tag{273}$$

Hence

$$\kappa_1 = -\frac{2i}{\alpha r \pi}\,[J_0(\alpha b)H_0(\alpha a) - J_0(\alpha a)H_0(\alpha b)] \tag{274}$$

and thus

$$v_1 = \frac{[J_0(\alpha r)H_0(\alpha a) - J_0(\alpha a)H_0(\alpha r)]\,[J_0(\alpha r')H_0(\alpha b) - H_0(\alpha r')J_0(\alpha b)]\,Q\alpha^2 r' \pi 1}{-2i[J_0(\alpha b)H_0(\alpha a) - J_0(\alpha a)H_0(\alpha b)]} \tag{275}$$

To apply the Heaviside expansion theorem to this it is known that the equation

$$Z = J_0(\alpha b)H_0(\alpha a) - J_0(\alpha a)H_0(\alpha b) = 0 \tag{276}$$

possesses an infinite number of real roots, and that to every positive root α_n there corresponds a negative root $-\alpha_n$. Also from (276)

$$\frac{J_0(\alpha a)}{J_0(\alpha b)} = \frac{H_0(\alpha a)}{H_0(\alpha b)} = \rho \text{ say.} \tag{277}$$

Then

$$\begin{aligned}\frac{dZ}{d\alpha} &= bJ_0'(\alpha b)\,H_0(\alpha a) + aJ_0(\alpha b)\,H_0'(\alpha a) - aJ_0'(\alpha a)\,H_0(\alpha b) - bJ_0(\alpha a)\,H_0'(\alpha b) \\ &= b\left[\frac{J_0'(\alpha b)\,H_0(\alpha b) - J_0(\alpha b)\,H_0'(\alpha b)}{\rho}\right] + a\rho[J_0(\alpha a)\,H_0'(\alpha a) - J_0'(\alpha a)\,H_0(\alpha a)] \\ &= -\frac{2ib}{\rho\pi\alpha b} + \frac{2ia\rho}{\pi\alpha a} = \frac{2i}{\pi\alpha}\left[\frac{\rho^2 - 1}{\rho}\right] \\ &= -\frac{2i}{\pi\alpha}\left[\frac{J_0^2(\alpha a) - J_0^2(\alpha b)}{J_0(\alpha a)\,J_0(\alpha b)}\right].\end{aligned} \tag{278}$$

Hence

$$\begin{aligned}v_1 \longleftrightarrow 2 \sum_1^\infty \frac{J_0(\alpha_n a)\,J_0(\alpha_n b)}{J_0^2(\alpha_n b) - J_0^2(\alpha_n a)} \cdot \frac{\alpha^2_n \pi^2 r'}{4}\,[J_0(\alpha_n r)\,H_0(\alpha_n a) - J_0(\alpha_n a)\,H_0(\alpha_n r)] \\ \times [J_0(\alpha_n r')\,H_0(\alpha_n b) - H_0(\alpha_n r')\,J_0(\alpha_n b)]\,e^{-\kappa\alpha^2_n t}\,1\end{aligned} \tag{279}$$

or $$v_1 \longleftrightarrow \frac{\pi^2}{2} \sum_1^\infty \alpha^2{}_n r' \frac{J_0^2(\alpha_n a)}{J_0^2(\alpha_n b) - J_0^2(\alpha_n a)} U_0(\alpha_n r)\, U_0(\alpha_n r')\, e^{-\kappa\alpha^2{}_n t}\, 1 \quad (280)$$

with a similar expression for v_2 where

$$U_0(\alpha r) = J_0(\alpha r)\, H_0(\alpha b) - J_0(\alpha b)\, H_0(\alpha r). \quad (281)$$

Now since (280) is the expression for G_B in this case, the temperature due to an arbitrary initial distribution $f(r)$ is given by

$$v \longleftrightarrow \frac{\pi^2}{2} \sum_1^\infty \alpha^2{}_n \frac{J_0^2(\alpha_n a)\, 1}{J_0^2(\alpha_n b) - J_0^2(\alpha_n a)} e^{-\kappa\alpha^2{}_n t}\, U_0(\alpha_n r) \int_a^b r f(r)\, U_0(\alpha_n r)\, dr \quad (282)$$

the summation being taken over the positive roots of

$$J_0(\alpha a)\, H_0(\alpha b) - J_0(\alpha b)\, H_0(\alpha a) = 0 \quad (283)$$

(See Carslaw, "Mathematical Theory of Heat," second edition, 1921, p. 129.)

In obtaining this solution it has been necessary to go through complicated manipulations in order to obtain the final result given by Carslaw. But the point to be borne in mind is that the actual solution is obtained when equation (269) is written down. It is true that this gives C_1 as a complicated expression involving r' and that it is finally reduced to a function of the J's and H's, but this is in the nature of getting the result into a convenient form adapted to numerical calculation, rather than an essential part of the problem. It can readily be expected that the expression whose roots will be required will be

$$J_0(\alpha b)\, H_0(\alpha a) - J_0(\alpha a)\, H_0(\alpha b) \quad (284)$$

since if we forget about the conditions at $r = r'$ and integrate from a to b, we obtain at once the condition that (284) must be equal to zero, and hence we are led to expect that (269) must be capable of simplification and contain (284) as a factor. But even if it had not been possible to effect this simplification, we could assign values of r_1, and find the roots of the coefficient of αC_1 in (269). Applying the Heaviside expansion theorem, etc., the same final numerical values would be obtained as if (280) were used. This would of course be a very much longer process and it is desirable, though not absolutely necessary, to effect whatever simplifications are possible.

GENERALIZATION OF EQUATION OF HEAT CONDUCTION.

The preceding solutions of problems in the conduction of heat have been made along the lines used by Heaviside in the solution of such problems. The method is easy to use when the problem is a one-dimensional one. When the problem is a two- or three-dimensional one, the above method has not been used successfully up to the present. The difficulty encountered is chiefly in writing down equations similar to the equation used above

$$-\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial x}=R\frac{dI}{dt} \tag{285}$$

on the successive integrations with respect to y and z.

We will now proceed to give a broader method which may be applied to problems in any number of dimensions. To do so it will be necessary to write the equation of conduction of heat in a form analogous to Poisson's equation in potential problems.

Laplace's equation is

$$\nabla^2 v=0 \tag{286}$$

and can only be used at points in the field at which there are no charges.

Poisson's equation is

$$\nabla^2 v=-4\pi\rho \tag{287}$$

which applies to points at which there are charges.

The ordinary equation of conduction of heat is

$$\nabla^2 v=\frac{1}{\kappa}\frac{\partial v}{\partial t} \tag{288}$$

and this equation applies only when no charges of heat are suddenly introduced into the field.

By referring to the derivation of the equation of conduction of heat (see, for instance, Carslaw, "Mathematical Theory of Heat," second edition, 1921, pp. 7 and 8) it will be found that considering an element of volume of the solid at the point P, namely, the rectangular parallelepiped with this point as centre, its edges being parallel to the coördinate axes, and of lengths $2dx$, $2dy$ and $2dz$, the rate at which the element is gaining heat from the flow through its faces is

$$-8\,dx\,dy\,dz\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right) \tag{289}$$

where

$$f_x=-\kappa\frac{\partial v}{\partial x},\ f_y=-\kappa\frac{\partial v}{\partial y},\ f_z=-\kappa\frac{\partial v}{\partial z} \tag{290}$$

The element of volume itself is gaining heat at the rate of

$$8\,dx\,dy\,dz\,c\rho\,\frac{\partial v}{\partial t} \tag{291}$$

If an instantaneous charge Q is placed at the point x', y', z', at the time $t=0$, then the rate at which the element of volume gains heat due to this charge is

$$Q\,\frac{dl}{dt} \tag{292}$$

since on integrating this

$$\int_{t_1-\varepsilon}^{t_1+\varepsilon} Q\,\frac{dl}{dt} = 0 \qquad t_1 \neq 0 \tag{293}$$

and

$$\int_{t_1-\varepsilon}^{t_1+\varepsilon} Q\,\frac{dl}{dt} = 1 \qquad t = 0 \tag{294}$$

Hereafter it will be assumed that l is the H-function given by

$$\left.\begin{aligned} l &= 0 \qquad t<0 \\ l &= 1 \qquad t>0 \\ 0 \leq l &\leq 1 \qquad t=0 \end{aligned}\right\} \tag{295}$$

Hence the equation of heat conduction now becomes

$$-\,8\,dx\,dy\,dz(-\,C\nabla^2 v) = 8\,dx\,dy\,dz\,C\rho\,\frac{\partial v}{\partial t} + Q\,\frac{dl}{dt} \tag{296}$$

or

$$\nabla^2 v = \frac{1}{\kappa}\,\frac{\partial v}{\partial t} + \frac{Q}{8C\,dx\,dy\,dz}\,\frac{dl}{dt} \tag{297}$$

But $\frac{Q}{8\,dx\,dy\,dz}$ is the heat introduced per unit volume, and is thus very analogous to the quantity ρ, the density in Poisson's equation. Of course when Q is finite, ρ is infinite. Thus the equation of conduction of heat may be written

$$\nabla^2 v = \frac{1}{\kappa}\,\frac{\partial v}{\partial t} + \frac{\rho}{C}\,\frac{dl}{dt} \tag{298}$$

We will now illustrate the application of this equation to two heat problems, first to find the Green's function G_B for a rod of length l whose ends are kept at zero temperature. This simple problem is taken first to show the full details of the solution. A simple extension to the two-dimensional case of the rectangle is then given. Following this the solution of Poisson's equation for a three-dimensional problem is worked out. The

general steps in the solution of the equation of heat conduction or Poisson's equation in orthogonal curvilinear coördinates are then pointed out.

THE GREEN'S FUNCTION FOR A ROD OF LENGTH l, ENDS x = 0 AND x = l KEPT AT ZERO TEMPERATURE.

The equations are

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\kappa}\frac{\partial v}{\partial t} + \frac{\rho}{c}\frac{dl}{dt} \tag{299}$$

$$v = 0 \quad \text{when } x = 0 \tag{300}$$

$$v = 0 \quad \text{when } x = l \tag{301}$$

$$\operatorname*{Lt}_{x = x_1 + 0} v = \operatorname*{Lt}_{x = x_1 - 0} v \tag{302}$$

Since ρ, regarded as a function of x, is zero when $x \neq x_1$, we may solve in the usual manner and find

$$v_1 = \sinh\left(x\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)A \qquad x < x_1 \tag{303}$$

and

$$v_2 = \sinh\left((l-x)\sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}\right)B \qquad x > x_1 \tag{304}$$

or

$$v_1 = \sinh x\alpha \,.\, A \tag{303a}$$

$$v_2 = \sinh (l - x)\alpha \,.\, B \tag{304a}$$

where $\alpha = \sqrt{\frac{1}{\kappa}\frac{\partial}{\partial t}}$ and A and B are functions of t alone.

Now consider the function X where

$$\left.\begin{aligned} X &= 0 & x &< x_1 \\ X &= 1 & x &> x_1 \\ 0 \leq X &\leq 1 & x &= x_1 \end{aligned}\right\} \tag{305}$$

This may be considered as a Heaviside unit function with respect to the coördinate x.

If we write

$$v = \sinh x\alpha \,.\, A(1 - X) + \sinh (l - x)\alpha \,.\, BX \tag{306}$$

this gives the same values for v_1 and v_2 as (303a) and (304a) since the second term is zero when $x < x_1$ and the first term is zero when $x > x_1$. We also have from (302)

$$\sinh x_1\alpha \,.\, A = \sinh (l - x_1)\alpha \,.\, B \tag{307}$$

or

$$B = \frac{\sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} \cdot A \tag{308}$$

Hence

$$v = \sinh x\alpha \,.\, A(1 - X) + \frac{\sinh (l - x)\,\alpha \,.\, \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} AX \qquad (309)$$

Let us now substitute this value of v in (299).

$$\frac{\partial v}{\partial x} = \alpha \cosh x\alpha \,.\, A\,(1 - X) - \sinh x\alpha \,.\, A \frac{\partial X}{\partial x} - \alpha \frac{\cosh (l - x)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} AX + \frac{\sinh (l - x)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} A \frac{\partial X}{\partial x} \qquad (310)$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 \sinh x\alpha \,.\, A\,(1 - X) - 2\alpha \cosh x\alpha A \frac{\partial X}{\partial x} - \sinh x\alpha A \frac{\partial^2 X}{\partial x^2} + \alpha^2 \frac{\sinh (l - x)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} AX - 2\alpha \frac{\cosh (l - x)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} A \frac{\partial X}{\partial x} + \frac{\sinh (l - x)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} A \frac{\partial^2 X}{\partial x^2} \qquad (311)$$

also from (309)

$$\frac{1}{\kappa} \frac{\partial v}{\partial t} + \frac{\rho}{c} \frac{dI}{dt} = \alpha^2 \sinh x\alpha \,.\, A\,(1 - X) + \alpha^2 \frac{\sinh (l - x)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} AX + \frac{\rho}{c} \frac{dI}{dt} \qquad (312)$$

since $\alpha^2 = \frac{1}{\kappa} \frac{\partial}{\partial t}$. Now when $x \neq x_1$, $t \neq 0$.

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\kappa} \frac{\partial v}{\partial t} \qquad (313)$$

since $\frac{\partial X}{\partial x} = 0$, $\frac{\partial^2 X}{\partial x^2} = 0$ and $\frac{dI}{dt} = 0$.

When $x = x_1$

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa} \frac{\partial v}{\partial t} = -2\alpha \cosh x_1\alpha A \frac{\partial X}{\partial x} - 2\alpha \frac{\cosh (l - x_1)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} A \frac{\partial X}{\partial x} \qquad (314)$$

the $\frac{\partial^2 X}{\partial x^2}$ terms disappear since their coefficients are equal but of opposite sign when $x = x_1$.

Hence at $x = x_1$

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa} \frac{\partial v}{\partial t} = -2\alpha \frac{\sinh (l - x_1)\,\alpha \cosh x_1\alpha + \cosh (l - x_1)\,\alpha \sinh x_1\alpha}{\sinh (l - x_1)\,\alpha} A \frac{\partial X}{\partial x} = -\frac{2\alpha \sinh l\alpha}{\sinh (l - x_1)\,\alpha} A \frac{\partial X}{\partial x} = \frac{\rho}{c} \frac{dI}{dt}. \qquad (315)$$

Hence integrating both sides of this last equation from $x_1 - \varepsilon$ to $x_1 + \varepsilon$ gives

$$A = -\frac{\sinh(l-x_1)\alpha}{2\alpha\sinh l\alpha}\frac{\int_{x_1-\varepsilon}^{x_1+\varepsilon}\rho dx}{c}\frac{dl}{dt}$$
$$= -\frac{Q}{c}\frac{\sinh(l-x_1)\alpha}{2\alpha\sinh l\alpha}\frac{dl}{dt} \tag{316}$$

since $\int_{x_1-\varepsilon}^{x_1+\varepsilon}\frac{\partial X}{\partial x}dx = 1$ numerically by definition of X, and $\int_{x_1-\varepsilon}^{x_1+\varepsilon}\rho dx = Q$ the amount of charge placed at the point $x = x_1$. Hence

$$v = -\frac{Q}{c}(1-X)\frac{\sinh x\alpha\sinh(l-x_1)\alpha}{2\alpha\sinh l\alpha}\frac{dl}{dt}$$
$$-\frac{Q}{c}X\frac{\sinh(l-x)\alpha\sinh x_1\alpha}{2\alpha\sinh l\alpha}\frac{dl}{dt}. \tag{317}$$

This result may be checked with that already obtained in the same problem, using different forms of equations in (246). Letting $h\to\infty$, which corresponds to writing $v = 0$ in (208) and (209), (246) becomes

$$v_1 = \frac{\sinh x\alpha\sinh(y-l)\alpha}{\alpha\sinh l\alpha}RQ\frac{dl}{dt} \tag{318}$$

and if $R = 1/c$, this is twice the coefficient of $(1-X)$ in (317). If the value of v_2 corresponding to v_1 given in (246) is taken, the result will be twice the coefficient of X in (317). This discrepancy always arises at this point when the method of determining the value of v given here is used. When the value of v from (309) is substituted in the original equation and x is put equal to x_1, the only term which remains to be equated to $\frac{\rho}{c}\frac{dl}{dt}$ is obtained from expressions derived from the values of v, both on the left and right of x_1, and since these are equal at the point $x = x_1$, it might be expected that the expression

$$-\frac{2\alpha\sinh l\alpha}{\sinh(l-x_1)\alpha}A\frac{\partial X}{\partial x}$$

in (315) would be double its proper value. If this method is retained, in every problem which is worked out using it, it is necessary to multiply the result obtained similar to (317) by 2. This will be done in future problems without any detailed reference being made to it. Probably the best way to avoid the introduction of the 2 is to substitute the values of v_1 and v_2 separately in the original equation of heat conduction and equate the results and finally arrive at (317). This, however, tends to destroy the

symmetry of the working out of the result, and thus the method used above will be retained and the 2 arbitrarily introduced.

Equation (317) may now be solved by the Heaviside expansion theorem or as an alternative by the method of elimination. The following method used by Heaviside in his electromagnetic theory before he gives his expansion theorem seems to present some advantages, as it allows us to follow out the reasoning, step by step. It really amounts to developing the expansion theorem for each particular problem. We have

$$\frac{1}{\alpha l \sinh \alpha l} \longleftrightarrow \frac{1}{\alpha^2 l^2} - \frac{2}{l^2\alpha^2 + \pi^2} + \frac{2}{l^2\alpha^2 + 2^2\pi^2} - \text{etc.} \tag{319}$$

Hence multiplying (317) by 2 as explained above

$$v \longleftrightarrow -\frac{Q}{lc}(1 - X)\sinh x\alpha \sinh (l - x_1)\alpha \left[\frac{1}{\alpha^2} - \frac{2}{\alpha^2 + \frac{\pi^2}{l^2}} + \cdots\right]\frac{dl}{dt}$$

$$-\frac{Q}{lc} X \sinh (l - x)\alpha \sinh x_1\alpha \left[\frac{1}{\alpha^2} - \frac{2}{\alpha^2 + \frac{\pi^2}{l^2}} + \cdots\right]\frac{dl}{dt} \tag{320}$$

Thus we have to determine quantities A_1, A_2, . . . A_n, . . . such that

$$A_n = \frac{1}{\alpha^2 + \frac{n^2\pi^2}{l^2}}\frac{dl}{dt} \tag{321}$$

or

$$\frac{1}{\kappa}\frac{dA_n}{dt} + \frac{n^2\pi^2}{l^2} A_n = \frac{dl}{dt} \tag{322}$$

We have solved similar equations before (199) *et seq.*, and may solve this by assuming

$$A_n = Ce^{-\kappa \frac{n^2\pi^2}{l^2} t} 1 \tag{323}$$

which gives

$$\frac{1}{\kappa}\frac{dA_n}{dt} = -C\frac{n^2\pi^2}{l^2} e^{-\kappa \frac{n^2\pi^2}{l^2} t} 1 + \frac{1}{\kappa} Ce^{-\kappa \frac{n^2\pi^2}{l^2}} \frac{dl}{dt} \tag{324}$$

Hence substituting in (322) and putting $t = 0$ since when $t \neq 0$ $\frac{dl}{dt} = 0$, we get

$$\frac{1}{\kappa} C\frac{dl}{dt} = \frac{dl}{dt} \tag{325}$$

or

$$C = \kappa \tag{326}$$

We thus replace each of the fractions in (320)

$$\frac{\frac{d1}{dt}}{\alpha^2+\frac{n^2\pi^2}{l^2}} \text{ by } A_n = \kappa e^{-\kappa\frac{n^2\pi^2}{l^2}t}1 \qquad n \neq 0 \tag{327}$$

and it will readily be found that operating on these by the operators $\sinh x\alpha \sinh(l-x_1)\alpha$ and $\sinh(l-x)\alpha \sinh x_1\alpha$, we get $\sin\frac{n\pi x}{l}\sin\frac{n\pi(l-x_1)}{l}A_n$ and $\sin\frac{n\pi(l-x)}{l}\sin\frac{n\pi x_1}{l}A_n$, respectively. There will in addition be terms involving $\frac{d1}{dt}$ + higher derivatives of $\frac{d1}{dt}$.

The first term in (320) gives $x(l-x_1)\frac{d1}{dt}$ + higher derivatives of $\frac{d1}{dt}$ when similar operations are carried out since

$$\underset{\alpha\to 0}{\text{Limit}}\ \frac{\sinh x\alpha \sinh(l-x_1)\alpha}{\alpha^2} \longleftrightarrow x(l-x_1). \tag{328}$$

Making these substitutions in (320), we get

$$\begin{aligned} v \longleftrightarrow & -\frac{2Q\kappa}{lc}(1-X)\left[x(l-x_1)\frac{d1}{dt}+\sum_{n=1}^{\infty}a_n\frac{d^n1}{dt^n}\right. \\ & \left.+\sum_{1}^{\infty}(-1)^n\sin\frac{n\pi x}{l}\sin\frac{n\pi(l-x_1)}{l}e^{-\kappa\frac{n^2\pi^2}{l^2}t}1\right] \\ & -\frac{2Q\kappa}{lc}X\left[x_1(l-x)\frac{d1}{dt}+\sum_{n=1}^{\infty}a_n'\frac{d^n1}{dt^n}\right. \\ & \left.+\sum_{1}^{\infty}(-1)^n\sin\frac{n\pi(l-x)}{l}\sin\frac{n\pi x_1}{l}e^{-\kappa\frac{n^2\pi^2}{l^2}t}1\right] \end{aligned} \tag{329}$$

When $t>0$, $\frac{d1}{dt}=0$ and $\frac{d^n1}{dt^n}=0$, hence these terms can be omitted. Also

$$\sin\frac{n\pi(l-x_1)}{l}=(-1)^{n-1}\sin\frac{n\pi x_1}{l}. \tag{330}$$

Thus

$$\begin{aligned} v \longleftrightarrow & \frac{2Q\kappa}{lc}(1-X)\sum_{1}^{\infty}\sin\frac{n\pi x}{l}\sin\frac{n\pi x_1}{l}e^{-\kappa\frac{n^2\pi^2}{l^2}t}1 \\ & +\frac{2Q\kappa}{lc}X\sum_{1}^{\infty}\sin\frac{n\pi x}{l}\sin\frac{n\pi x_1}{l}e^{-\kappa\frac{n^2\pi^2}{l^2}t}1 \end{aligned} \tag{331}$$

and it is evident that this may be written

$$v \longleftrightarrow \frac{2Qk}{lc} \sum_1^\infty \sin \frac{n\pi x}{l} \sin \frac{n\pi x_1}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} 1 \tag{332}$$

It may be well to point out here the difference in the way in which this result satisfies the original equation of conduction compared with the ordinary solution in the Theory of Functions which omits the 1. We have

$$\frac{\partial^2 v}{\partial x^2} \longleftrightarrow - \frac{2QK}{lc} \sum_1^\infty \frac{n^2\pi^2}{l^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi x_1}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} 1 \tag{333}$$

$$- \frac{1}{\kappa} \frac{\partial v}{\partial t} \longleftrightarrow + \frac{2QK}{lc} \sum_1^\infty \frac{n^2\pi^2}{l^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi x_1}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} 1$$

$$- \frac{2Q}{lc} \sum_1^\infty \sin \frac{n\pi x}{l} \sin \frac{n\pi x_1}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} \frac{d1}{dt} \tag{334}$$

Therefore

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa} \frac{\partial v}{\partial t} \longleftrightarrow \frac{2Q}{lc} \Sigma \sin \frac{n\pi x}{l} \sin \frac{n\pi x_1}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} \frac{d1}{dt}$$

$$\longleftrightarrow \frac{\rho}{c} \frac{d1}{dt}. \tag{335}$$

Hence we must have when $t = 0$ (since when $t \neq 0$, $\frac{d1}{dt} = 0$)

$$\frac{2Q}{l} \int_{x_1-\varepsilon}^{x_1+\varepsilon} \Sigma \sin \frac{n\pi x}{l} \sin \frac{n\pi x_1}{l} dx \longleftrightarrow \int_{x_1+\varepsilon}^{x_1+\varepsilon} \rho dx \longleftrightarrow Q \tag{336}$$

or

$$\frac{2}{l} \int_{x_1-\varepsilon}^{x_1+\varepsilon} \Sigma \sin \frac{n\pi x}{l} \sin \frac{n\pi x_1}{l} dx \longleftrightarrow 1 \tag{337}$$

a result which has already been shown to be true in the theory of H-functions (see (118)).

THE GREEN'S FUNCTION FOR A RECTANGULAR LAMINA WHOSE EDGES ARE KEPT AT ZERO TEMPERATURE.

The equations of conduction are

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{\kappa} \frac{\partial v}{\partial t} + \frac{\rho}{c} \frac{d1}{dt} \tag{338}$$

$$v = 0 \qquad x = 0 \tag{339}$$

$$v = 0 \qquad x = l \tag{340}$$

$$v = 0 \qquad y = 0 \tag{341}$$

$$y = 0 \qquad y = m \tag{342}$$

$$\rho = 0 \text{ when } x \neq x_1 \text{ and } y \neq y_1 \tag{343}$$

and $$\int_{x_1-\varepsilon}^{x_1+\varepsilon}\int_{y_1-\varepsilon}^{y_1+\varepsilon}\rho dxdy = Q \text{ when } x = x_1 \text{ and } y = y_1,\ t = 0 \qquad (344)$$

Proceeding as before, we get

$$v = \sinh x\alpha \,.\, A(1-X) + \frac{\sinh(l-x)\alpha \sinh x_1\alpha}{\sinh(l-x_1)\alpha} AX \qquad (345)$$

where

$$\alpha^2 = \frac{1}{\kappa}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} \qquad (346)$$

and

$$A = -\frac{\sinh(l-x_1)\alpha}{\alpha \sinh l\alpha}\frac{\int_{x_1-\varepsilon}^{x_1+\varepsilon}\rho dx}{c}\frac{dl}{dt} \qquad (347)$$

this being twice the value of A actually obtained as explained before. We thus have to solve the equations

$$A_n = \frac{1}{\alpha^2 + \frac{n^2\pi^2}{l^2}}\int_{x_1-\varepsilon}^{x_1+\varepsilon}\rho dx\frac{dl}{dt} \qquad (348)$$

where A_n is now a function of y and t. We will solve this equation first with respect to y. Inserting the value of α from (346), it becomes

$$\frac{1}{\kappa}\frac{\partial A_n}{\partial t} - \frac{\partial^2 A_n}{\partial y^2} + \frac{n^2\pi^2}{l^2}A_n = \int_{x_1-\varepsilon}^{x_1+\varepsilon}\rho dx\frac{dl}{dt} \qquad (349)$$

Since when the solution is completed by the method of elimination, it will be of the form

$$v = \sum_{n=1}^{\infty} f_n(x)\cdot A_n$$

it is evident that if we take

$$A_n = 0 \qquad y = 0 \qquad (350)$$

$$A_n = 0 \qquad y = m \qquad (351)$$

these terminal conditions for A_n are consistent with the terminal conditions (341) and (342) for v. They will at least give us one solution though we have not shown that there are no other solutions.

Using (350) and (351) and noting that the right-hand side of (349) is zero when $y \neq y_1$, we have a set of equations to solve which are similar to those we had to solve before with respect to x. Their solution thus becomes

$$A_n = \sinh\beta y \,.\, B(1-Y) + \frac{\sinh(m-y)\beta \sinh y_1\beta}{\sinh(m-y_1)\beta} BY \qquad (352)$$

where
$$B = \frac{\sinh(m-y_1)\beta}{\beta \sinh m\beta} \int_{x_1-\varepsilon}^{x_1+\varepsilon} \int_{y_1-\varepsilon}^{y_1+\varepsilon} \frac{\rho}{c}\, dx\, dy\, \frac{dI}{dt}$$
$$= \frac{Q}{c} \frac{\sinh(m-y_1)\beta}{\beta \sinh m\beta} \frac{dI}{dt} \tag{353}$$

where
$$\beta^2 = \frac{1}{\kappa}\frac{\partial}{\partial t} + \frac{n^2\pi^2}{l^2} \tag{354}$$

and Y is the Heaviside unit function with respect to y similar to that defined in (305) with respect to x. Expanding (352) as in the one-dimensional case,

$$A_n \longleftrightarrow \frac{Q}{mc}(1-Y)\sinh\beta y \sinh\beta(m-y_1)\left[\frac{1}{\beta^2} - \frac{2}{\beta^2+\frac{\pi^2}{m^2}} + \text{etc.}\right]\frac{dI}{dt}$$
$$+ \frac{Q}{mc} Y \sinh\beta(m-y)\sinh\beta y_1 \left[\frac{1}{\beta^2} - \frac{2}{\beta^2+\frac{\pi^2}{m^2}} + \text{etc.}\right]\frac{dI}{dt} \tag{355}$$

Thus we have to find quantities B_1, B_2, B_q, etc., such that
$$B_q = \frac{1}{\beta^2 + \frac{q^2\pi^2}{m^2}} \frac{dI}{dt} \tag{356}$$

or
$$\frac{1}{\kappa}\frac{dB_q}{dt} + \left(\frac{n^2\pi^2}{l^2} + \frac{q^2\pi^2}{m^2}\right)B_q = \frac{dI}{dt} \tag{357}$$

Hence as before in (322)
$$B_q = \kappa e^{-\kappa\left(\frac{n^2\pi^2}{l^2} + \frac{q^2\pi^2}{m^2}\right)t} 1 \tag{358}$$

And thus we have as before in the one-dimensional case
$$A_n \longleftrightarrow \frac{2\kappa Q}{cm} \Sigma \sin\frac{q\pi y}{m} \sin\frac{q\pi y_1}{m}\, e^{-\kappa\left(\frac{n^2\pi^2}{l^2} + \frac{q^2\pi^2}{m^2}\right)t} 1 \tag{359}$$

and substituting these values of A_n in the equation for v, we finally get when $t \neq 0$
$$v \longleftrightarrow \frac{4\kappa Q}{cml} \Sigma\Sigma \sin\frac{m\pi x}{l} \sin\frac{n\pi x_1}{l} \sin\frac{q\pi y}{m} \sin\frac{q\pi y_1}{m}\, e^{-\kappa\left(\frac{n^2\pi^2}{l^2} + \frac{q^2\pi^2}{m^2}\right)t} 1 \tag{360}$$

as the Green's function G for the space given by (338) to (342).

THE GREEN'S FUNCTION IN POTENTIAL FOR THE SPACE BOUNDED BY TWO PARALLEL PLANES, TWO AXIAL PLANES AND TWO CYLINDERS.

The equations to be solved are

$$\frac{\partial^2 v}{d\rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial z^2} = -4\pi\sigma \tag{361}$$

$$v = 0 \qquad z = 0 \tag{362}$$

$$v = 0 \qquad z = c \tag{363}$$

$$v = 0 \qquad \phi = 0 \tag{364}$$

$$v = 0 \qquad \phi = \delta \tag{365}$$

$$v = 0 \qquad \rho = a \tag{366}$$

$$v = 0 \qquad \rho = b \tag{367}$$

$$\sigma = 0 \qquad z \neq z_1 \qquad \phi \neq \phi_1 \qquad \rho \neq \rho_1 \tag{368}$$

$$\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\int_{\rho_1-\varepsilon}^{\rho_1+\varepsilon} \sigma\rho dz d\phi d\rho = Q \tag{369}$$

$$\operatorname*{Lt}_{z=z_1+0} v = \operatorname*{Lt}_{z=z_1-0} v \tag{370}$$

$$\operatorname*{Lt}_{\phi=\phi_1+0} v = \operatorname*{Lt}_{\phi=\phi_1-0} v \tag{371}$$

$$\operatorname*{Lt}_{\rho=\rho_1+0} v = \operatorname*{Lt}_{\rho=\rho_1-0} v \tag{372}$$

Integrating first with respect to z, we get, as before,

$$v = \sinh z\alpha \cdot A\,(1 - Z) + \frac{\sinh (c - z)\,\alpha \sinh z_1\alpha}{\sinh (c - z_1)\,\alpha} AZ \tag{373}$$

where

$$\alpha^2 = -\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right) \tag{374}$$

and

$$\begin{aligned} Z &= 0 \qquad & z &< z_1 \\ Z &= 1 \qquad & z &> z_1 \\ 0 \leq Z &\leq 1 \qquad & z &= z_1 \end{aligned} \tag{375}$$

Substituting (373) in (361) and putting $z = z_1$, we get

$$-4\pi\sigma = -\frac{2\alpha \sinh c\alpha}{\sinh (c - z_1)\alpha} A \frac{dZ}{dz} \tag{376}$$

Omitting the 2 from the right-hand side of (376) for reasons already explained and integrating this equation from $z_1 - \varepsilon$ to $z_1 + \varepsilon$,

$$A = \frac{4\pi \sinh (c - z_1)\alpha}{\alpha \sinh c\alpha} \int_{z_1-\varepsilon}^{z_1+\varepsilon} \sigma dz \tag{377}$$

and as before, this may be written

$$A \longleftrightarrow \frac{4\pi \sinh (c - z_1)\alpha}{c}\left[\frac{1}{\alpha^2} - \frac{2}{\alpha^2 + \frac{\pi^2}{c^2}} + \cdots\right]\int_{z_1-\varepsilon}^{z_1+\varepsilon} \sigma dz \tag{378}$$

To solve this we have to determine quantities A_1, A_2, . . . A_p, . . . such that

$$A_p = \frac{1}{\alpha^2 + \frac{p^2\pi^2}{c^2}}\int_{z_1-\varepsilon}^{z_1+\varepsilon} \sigma dz \tag{379}$$

or substituting the value of α

$$-\frac{\partial^2 A_p}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial A_p}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 A_p}{\partial \phi^2} + \frac{p^2\pi^2}{c^2} A_p = \int_{z_1-\varepsilon}^{z_1+\varepsilon} \sigma dz \tag{380}$$

To integrate these equations we may take as terminal conditions

$$\left.\begin{aligned} A_p &= 0 \qquad \phi = 0 \\ A_p &= 0 \qquad \phi = \delta \end{aligned}\right\} \tag{381}$$

$$\operatorname*{Lt}_{\phi = \phi_1 + 0} A_p = \operatorname*{Lt}_{\phi = \phi_1 - 0} A_p \tag{382}$$

It has already been shown (381) and (382) are consistent with (364), (365) and (371). We thus get

$$A_p = \sinh \phi\beta \,.\, B(1 - \Phi) + \frac{\sinh (\delta - \phi)\beta \sinh \phi_1\beta}{\sinh (\delta - \phi_1)\beta} B\Phi \tag{383}$$

where

$$\beta^2 = \frac{p^2\pi^2}{c^2}\rho^2 - \rho^2\frac{\partial^2}{\partial \rho^2} - \rho\frac{\partial}{\partial \rho} \tag{384}$$

and

$$\left.\begin{aligned} \Phi &= 0 \qquad \phi < \phi_1 \\ \Phi &= 1 \qquad \phi > \phi_1 \\ 0 \leq \Phi &\leq 1 \qquad \phi = \phi_1 \end{aligned}\right\} \tag{385}$$

Substituting (383) in (380), we get

$$-\rho^2\int_{z_1-\varepsilon}^{z_1+\varepsilon} \sigma\, dz = -\frac{2\beta \sinh \delta\beta}{\sinh (\delta - \phi_1)\beta} B\frac{d\Phi}{d\phi} \tag{386}$$

omitting the 2 as usual and integrating from $\phi_1 - \varepsilon$ to $\phi_1 + \varepsilon$, we get

$$\frac{\beta \sinh \delta\beta}{\sinh (\delta - \phi_1)\beta} B = \rho^2\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon} \sigma\, dz\, d\phi \tag{387}$$

or

$$B = \frac{\sinh (\delta - \phi_1)\beta}{\beta \sinh \delta\beta}\rho^2\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon} \sigma\, dz\, d\phi \tag{388}$$

which may be written

$$B \longleftrightarrow \frac{\sinh(\delta-\phi_1)\beta}{\delta}\left[\frac{\rho^2}{\beta^2}-\frac{2\rho^2}{\beta^2+\frac{\pi^2}{\delta^2}}+\text{etc.}\right]\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sigma\,dz\,d\phi \tag{389}$$

To solve this equation it is necessary to determine functions $R_1,\ R_2,\ R_3,\ \ldots\ R_m\ \ldots$ such that

$$R_m = \frac{\rho^2}{\beta^2+\frac{m^2\pi^2}{\delta^2}}\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sigma\,dz\,d\phi \tag{390}$$

or

$$\frac{\partial^2 R_m}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial R_m}{\partial\rho}-\left(\frac{\rho^2\pi^2}{c^2}+\frac{m^2\pi^2}{\delta^2}\frac{1}{\rho^2}\right)R_m = -\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sigma\,dz\,d\phi \tag{391}$$

writing $\frac{p\pi}{c}=\lambda$ and $\frac{m\pi}{\delta}=s$, the solution of this equation is

$$R_m = CI_s(\lambda\rho)+C'K_s(\lambda\rho) \tag{392}$$

For terminal conditions we will take in the usual manner

$$\left.\begin{aligned} R_m &= 0 && \rho = a\\ R_m &= 0 && \rho = b\\ \operatorname*{Lt}_{\rho=\rho_1+0} R_m &= \operatorname*{Lt}_{\rho=\rho_1-0} R_m \end{aligned}\right\} \tag{393}$$

which are consistent with (366), (367) and (372)

Thus

$$C_1 I_s(\lambda a)+C_2 K_s(\lambda a) = 0 \tag{394}$$

and

$$C_2 I_s(\lambda b)+C_1 K_s(\lambda b) = 0 \tag{395}$$

Hence we find

$$\begin{aligned} R_m = &\left[I_s(\lambda\rho)-\frac{I_s(\lambda a)\,K_s(\lambda\rho)}{K_s(\lambda a)}\right]C_1(1-R)\\ &+\left[I_s(\lambda\rho)-\frac{I_s(\lambda b)\,K_s(\lambda\rho)}{K_s(\lambda b)}\right]C_2 R \end{aligned} \tag{396}$$

where as usual

$$\left.\begin{aligned} R &= 0 && \rho<\rho_1\\ R &= 1 && \rho>\rho_1\\ 0\leq R &\leq 1 && \rho=\rho_1 \end{aligned}\right\} \tag{397}$$

also since

$$\operatorname*{Lt}_{\rho=\rho_1-0} R_m = \operatorname*{Lt}_{\rho=\rho_1+0} R_m \tag{398}$$

$$C_1\left[\frac{I_s(\lambda\rho_1)\,K_s(\lambda a)-I_s(\lambda a)\,K_s(\lambda\rho_1)}{K_s(\lambda a)}\right] = C_2\left[\frac{I_s(\lambda\rho_1)\,K_s(\lambda b)-I_s(\lambda b)\,K_s(\lambda\rho_1)}{K_s(\lambda b)}\right] \tag{399}$$

Differentiating (396)

$$\frac{\partial R_m}{\partial \rho} = \lambda\left[I_s'(\lambda\rho) - \frac{I_s(\lambda a)\,K_s'(\lambda\rho)}{K_s(\lambda a)}\right]C_1(1-R) - \left[I_s(\lambda\rho) - \frac{I_s(\lambda a)\,K_s(\lambda\rho)}{K_s(\lambda a)}\right]C_1\frac{dR}{d\rho}$$
$$+\lambda\left[I_s'(\lambda\rho) - \frac{I_s(\lambda b)\,K_s'(\lambda\rho)}{K_s(\lambda b)}\right]C_2R + \left[I_s(\lambda\rho) - \frac{I_s(\lambda b)\,K_s(\lambda\rho)}{K_s(\lambda b)}\right]C_2\frac{dR}{d\rho} \quad (400)$$

and

$$\frac{\partial^2 R_m}{\partial \rho^2} = \lambda^2\left[I_s''(\lambda\rho) - \frac{I_s(\lambda a)\,K_s''(\lambda\rho)}{K_s(\lambda a)}\right]C_1(1-R) - 2\lambda\left[I_s'(\lambda\rho) - \frac{I_s(\lambda a)\,K_s'(\lambda\rho)}{K_s(\lambda a)}\right]C_1\frac{dR}{d\rho}$$
$$-\left[I_s(\lambda\rho) - \frac{I_s(\lambda a)\,K_s(\lambda\rho)}{K_s(\lambda a)}\right]C_1\frac{d^2R}{d\rho^2} + \lambda^2\left[I_s''(\lambda\rho) - \frac{I_s(\lambda b)\,K_s''(\lambda\rho)}{K_s(\lambda b)}\right]C_2R$$
$$+2\lambda\left[I_s'(\lambda\rho) - \frac{I_s(\lambda b)\,K_s'(\lambda\rho)}{K_s(\lambda b)}\right]C_2\frac{dR}{d\rho} + \left[I_s(\lambda\rho) - \frac{I_s(\lambda b)\,K_s(\lambda\rho)}{K_s(\lambda b)}\right]C_2\frac{d^2R}{d\rho^2}. \quad (401)$$

Substituting these values in (391), we get, using (399) and noticing that the coefficients of $\frac{d^2R}{d\rho^2}$ are equal but of opposite sign when $\rho = \rho_1$, and also omitting the factor 2 in the usual manner

$$-\left[\frac{I_s'(\lambda\rho_1)\,K_s(\lambda a) - I_s(\lambda a)\,K_s'(\lambda\rho_1)}{K_s(\lambda a)}\right]C_1\lambda\frac{dR}{d\rho}$$
$$+\left[\frac{I_s'(\lambda\rho_1)\,K_s(\lambda b) - I_s(\lambda b)\,K_s'(\lambda\rho_1)}{K_s(\lambda b)}\right]C_2\lambda\frac{dR}{d\rho} = -\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sigma\,dz\,d\phi \quad (402)$$

using (399), the left-hand side of this equation becomes

$$C_1\lambda\frac{dR}{d\rho}\frac{[I_s'(\lambda\rho_1)\,K_s(\lambda b) - I_s(\lambda b)\,K_s'(\lambda\rho)]\,[I_s(\lambda\rho_1)\,K_s(\lambda a) - I_s(\lambda a)\,K_s(\lambda\rho_1)]}{K_s(\lambda a)\,[I_s(\lambda\rho_1)\,K_s(\lambda b) - I_s(\lambda b)\,K_s(\lambda\rho_1)]}$$
$$-C_1\lambda\frac{dR}{d\rho}\frac{[I_s'(\lambda\rho_1)\,K_s(\lambda a) - I_s(\lambda a)\,K_s'(\lambda\rho_1)]}{K_s(\lambda a)}$$

multiplying out and rearranging, this becomes

$$C_1\lambda\frac{dR}{d\rho}\frac{[I_s(\lambda a)\,K_s(\lambda b) - K_s(\lambda a)\,I_s(\lambda b)]\,[I_s(\lambda\rho_1)\,K_s'(\lambda\rho_1) - I_s(\lambda\rho_1)\,K_s(\lambda\rho_1)]}{K_s(\lambda a)\,[I_s(\lambda\rho_1)\,K_s(\lambda b) - I_s(\lambda b)\,K_s(\lambda\rho_1)]}$$

but

$$I_s(\lambda\rho_1)\,K_s'(\lambda\rho_1) - I_s'(\lambda\rho_1)\,K_s(\lambda\rho_1) = -\frac{1}{\lambda\rho_1} \quad (403)$$

(See, for instance, Gray and Matthews, "Bessel Functions," second edition, 1922, p. 25.)

Hence

$$C_1\frac{dR}{d\rho}\frac{I_s(\lambda a)\,K_s(\lambda b) - K_s(\lambda a)\,I_s(\lambda b)}{K_s(\lambda a)\,[I_s(\lambda\rho_1)\,K_s(\lambda b) - I_s(\lambda b)\,K_s(\lambda\rho_1)]} = \int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sigma\rho_1\,dz\,d\phi \quad (404)$$

and integrating both sides of this equation from $\rho_1 - \varepsilon$ to $\rho_1 + \varepsilon$, using (369), we get

$$C_1 \frac{I_s(\lambda a) K_s(\lambda b) - K_s(\lambda a) I_s(\lambda b)}{K_s(\lambda a)[I_s(\lambda\rho_1) K_s(\lambda b) - I_s(\lambda b) K_s(\lambda\rho_1)]} = Q. \tag{405}$$

Hence

$$R_m = Q(1 - R) \frac{[I_s(\lambda\rho) K_s(\lambda a) - I_s(\lambda a) K_s(\lambda\rho)]\,[I_s(\lambda\rho_1) K_s(\lambda b) - I_s(\lambda b) K_s(\lambda\rho_1)]}{I_s(\lambda a) K_s(\lambda b) - K_s(\lambda a) I_s(\lambda b)}$$
$$+ QR \frac{[I_s(\lambda\rho) K_s(\lambda b) - I_s(\lambda b) K_s(\lambda\rho)]\;\;[I_s(\lambda\rho_1) K_s(\lambda a) - I_s(\lambda a) K_s(\lambda\rho_1)]}{I_s(\lambda a) K_s(\lambda b) - K_s(\lambda a) I_s(\lambda b)} \tag{406}$$

This gives

$$A_p \longleftrightarrow \frac{2}{\delta} \Sigma \sin \frac{m\pi\phi}{\delta} \sin \frac{m\pi\phi_1}{\delta} R_m \tag{407}$$

and finally

$$v \longleftrightarrow \frac{16\pi}{c\delta} \sum_1^\infty \sum_1^\infty \sin \frac{p\pi z}{c} \sin \frac{p\pi z_1}{c} \sin \frac{m\pi\phi}{\delta} \sin \frac{m\pi\phi_1}{\delta} R_m \tag{408}$$

which may be compared with the result given by Gray and Matthews' "Bessel Functions" (second edition, 1922, p. 109). On p. 103 of this work Gray and Matthews point out that three such solutions can be obtained for the Green's function. These have been named by Dougall the z, ρ and ϕ forms. The one which has been worked out here is the ρ form, and it will be noticed that it contains the Heaviside unit function R with respect to ρ, thus enabling one solution to be given for the whole range $a \leq \rho \leq b$ instead of the two solutions usually given, one for $\rho < \rho_1$ and the other for $\rho > \rho_1$. It will be found that if the integrations are made with respect to z and ρ first, the ϕ form will be obtained, and if with respect to ρ and ϕ first the z form will be obtained. The work in these two latter cases follows along exactly the same lines as above for the ρ form, and it is unnecessary to go into it here. It may be well to add that the reason for not considering terms such as $\frac{1}{\alpha^2}$ in (378) is that all such terms give zero except at the point where the charge is, as may be seen from the results carried through for similar terms in (329).

A simple way to check (408) is to proceed as follows. Let v_1 be the value of v when $\rho < \rho_1$ and v_2 the value when $\rho > \rho_1$. Then

$$-\frac{\partial v_1}{\partial \rho} + \frac{\partial v_2}{\partial \rho} \longleftrightarrow \frac{16\pi Q}{c\delta} \sum_1^\infty \sum_1^\infty \sin \frac{p\pi z}{c} \sin \frac{p\pi z_1}{c} \sin \frac{m\pi\phi}{\delta} \sin \frac{m\pi\phi_1}{\delta}$$

$$\times\left(-\frac{\partial R_{m1}}{\partial \rho}+\frac{\partial R_{m2}}{\partial \rho}\right) \qquad (408a)$$

$$\longleftrightarrow 4\pi\sigma_1 \qquad (\rho=\rho_1)$$

where σ_1 is regarded as the surface density on a small element of surface perpendicular to the radius, and R_{m1} and R_{m2} are the coefficients of $(1-R)$ and R, respectively, in (406).

Evaluating this, after some rearrangement, we find

$$-\frac{\partial R_{m1}}{\partial \rho}+\frac{\partial R_{m2}}{\partial \rho}=\lambda\left[J_s'(\lambda\rho_1)\,K_s(\lambda\rho_1)-I_s(\lambda\rho_1)\,K_s'(\lambda\rho_1)\right]$$

$$=\frac{\lambda}{\lambda\rho_1}=\frac{1}{\rho_1} \qquad (408b)$$

Hence

$$-\frac{\partial v_1}{\partial \rho}+\frac{\partial v_2}{\partial \rho}\longleftrightarrow\frac{16\pi Q}{\rho_1 c\delta}\sum_1^\infty\sum_1^\infty \sin\frac{p\pi z}{c}\sin\frac{p\pi z_1}{c}\sin\frac{m\pi\phi}{\delta}\sin\frac{m\pi\phi_1}{\delta}$$

$$\longleftrightarrow 4\pi\sigma_1 \qquad (408c)$$

By taking the integral of σ_1 over a small element of surface at $\rho=\rho_1$, $z=z_1$, $\phi=\phi_1$, we should obtain the charge Q at this point.

$$\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sigma_1\,dz\,\rho_1\,d\phi\longleftrightarrow\frac{4Q}{c\delta}\sum_1^\infty\int_{z_1-\varepsilon}^{z_1+\varepsilon}\sin\frac{p\pi z}{c}\sin\frac{p\pi z_1}{c}\,dz_1$$

$$\times\sum_1^\infty\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sin\frac{m\pi\phi}{\delta}\sin\frac{m\pi\phi_1}{\delta}\,d\phi \qquad (408d)$$

It has already been shown that in the theory of H-functions the sum of the z integrals converges to $\frac{c}{2}$ and the sum of the ϕ integrals converges to $\frac{\delta}{2}$. Hence

$$\int_{z_1-\varepsilon}^{z_1+\varepsilon}\int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon}\sigma_1\,dz\,\rho_1\,d\phi\longleftrightarrow Q. \qquad (408e)$$

If this integral is taken at any other point than (z_1, ϕ_1, ρ_1) the result obtained is zero.

LAPLACE'S EQUATION IN ORTHOGONAL CURVILINEAR COÖRDINATES.

Although the method of procedure should be clear in this case from the preceding examples, it may be well to consider the first stages of the solution in the general case. For the derivation of Laplace's equation in these coördinates reference may be made to works dealing with the subject, as, for instance, Byerly, "Fourier Series," Chap. VIII, pp. 238 *et seq.* On p. 239 he gives the following relations:

$$\nabla^2 v=h_1h_2h_3\left[D_\lambda\left(\frac{h_1}{h_2h_3}D_\lambda v\right)+D_\mu\left(\frac{h_2}{h_3h_1}D_\mu v\right)+D_\nu\left(\frac{h_3}{h_2h_1}D_\nu v\right)\right] \qquad (409)$$

where

$$\left.\begin{aligned} F_1(x,y,z) &= \lambda \\ F_2(x,y,z) &= \mu \\ F_3(x,y,z) &= \nu \end{aligned}\right\} \tag{410}$$

and

$$1/h_1^2 = (D_\lambda x)^2 + (D_\lambda y)^2 + (D_\lambda z)^2 \tag{411}$$

$$1/h_2^2 = (D_\mu x)^2 + (D_\mu y)^2 + (D_\mu z)^2 \tag{412}$$

$$1/h_3^2 = (D_\nu x)^2 + (D_\nu y)^2 + (D_\nu z)^2 \tag{413}$$

For potential problems Poisson's equation becomes

$$D_\lambda(\phi_1 D_\lambda v) + D_\mu(\phi_2 D_\mu v) + D_\nu(\phi_3 D_\nu v) = -\frac{4\pi\sigma}{h_1 h_2 h_3} \tag{414}$$

and the equation of conduction of heat becomes

$$D_\lambda(\phi_1 D_\lambda v) + D_\mu(\phi_2 D_\mu v) + D_\nu(\phi_3 D_\nu v) = \frac{1}{\kappa h_1 h_2 h_3}\frac{\partial v}{\partial t} + \frac{\sigma}{c h_1 h_2 h_3}\frac{dl}{dt} \tag{415}$$

where

$$\left.\begin{aligned} \phi_1 &= \frac{h_1}{h_2 h_3} \\ \phi_2 &= \frac{h_2}{h_3 h_1} \\ \phi_3 &= \frac{h_3}{h_1 h_2} \end{aligned}\right\} \tag{416}$$

and σ is the volume density of the charge

$$\sigma = \frac{Q}{dx\,dy\,dz} = \frac{Q h_1 h_2 h_3}{d\lambda\, d\mu\, d\nu} \tag{417}$$

(See Byerly, *loc. cit.*)

ϕ_2 and ϕ_3 may, of course, be functions of λ. Let us regard for the moment the operators D_μ, D_ν and $\frac{\partial}{\partial t}$ as mere symbols, then (414) and (415) are simply equations of the second order with respect to λ. Let $F_1(\lambda)$ and $F_2(\lambda)$ be two distinct and independent solutions of one of these equations when $\sigma = 0$. Then if the terminal conditions with respect to λ are

$$v = 0 \qquad \lambda = \lambda_1 \tag{418}$$

$$v = 0 \qquad \lambda = \lambda_2 \tag{419}$$

and if the charge is placed at the point λ_1, the solution obtained in a manner similar to that used before is

$$v = \frac{F_2(\lambda_1)F_1(\lambda) - F_1(\lambda_1)F_2(\lambda)}{F_2(\lambda_1)} A(1-\Lambda) + \frac{F_2(\lambda_2)F_1(\lambda) - F_1(\lambda_2)F_2(\lambda)}{F_2(\lambda_2)} B\Lambda \tag{420}$$

where
$$B = \frac{F_2(\lambda_3)\,F_2(\lambda_1)\,F_1(\lambda') - F_1(\lambda_1)\,F_2(\lambda')}{F_2(\lambda_1)\,F_2(\lambda_2)\,F_1(\lambda') - F_1(\lambda_2)\,F_2(\lambda')}\,A \tag{421}$$

and
$$\left.\begin{aligned} \Lambda &= 0 & \lambda &< \lambda' \\ \Lambda &= 1 & \lambda &> \lambda' \\ 0 \leq \Lambda &\leq 1 & \lambda &= \lambda' \end{aligned}\right\} \tag{422}$$

and (421) follows from
$$\underset{\lambda=\lambda'+0}{\mathrm{Lt}}\; v = \underset{\lambda=\lambda'-0}{\mathrm{Lt}}\; v \tag{423}$$

Substituting this value of v in the original equation (let us consider (414)), we find after some simplification and omitting the 2 which arises and which has been referred to before
$$\frac{[F_1(\lambda')\,F_2'(\lambda') - F_1'(\lambda')\,F_2(\lambda')]\,[F_1(\lambda_1)\,F_2(\lambda_2) - F_1(\lambda_2)\,F_2(\lambda_1)]}{F_2(\lambda_1)\,[F_2(\lambda_2)\,F_1(\lambda') - F_1(\lambda_2)\,F_2(\lambda')]}\,A\,\frac{d\Lambda}{d\lambda} = -\frac{4\pi\sigma}{h_1 h_2 h_3} \tag{424}$$

and on integrating with respect to λ from $\lambda' - \varepsilon$ to $\lambda' + \varepsilon$
$$\frac{[F_1(\lambda')\,F_2'(\lambda') - F_1'(\lambda')\,F_2(\lambda')]\,[F_1(\lambda_1)\,F_2(\lambda_2) - F_1(\lambda_2)\,F_2(\lambda_1)]}{F_2(\lambda_1)\,[F_2(\lambda_2)\,F_1(\lambda') - F_1(\lambda_2)\,F_2(\lambda')]}\,A = -4\pi\int_{\lambda'-\varepsilon}^{\lambda'+\varepsilon}\frac{\sigma}{h_1 h_2 h_3}\,d\lambda \tag{425}$$

The coefficient of A can always be simplified. Since $F_1(\lambda)$ and $F_2(\lambda)$ are solutions of (414) when $\sigma = 0$, we have
$$D_\lambda\,[\phi_1 D_\lambda\,F_1(\lambda)] = -\,[D_\mu\,\phi_2 D_\mu + D_\nu\,\phi_3 D_\nu]\,F_1(\lambda) \tag{426}$$

and
$$D_\lambda\,[\phi_1 D_\lambda\,F_2(\lambda)] = -\,[D_\mu\,\phi_2 D_\mu + D_\nu\,\phi_3 D_\nu]\,F_2(\lambda) \tag{427}$$

multiplying (426) by $F_2(\lambda)$ and (427) by $F_1(\lambda)$ and subtracting
$$F_2(\lambda)\,D_\lambda\,[\phi_1 D_\lambda\,F_2(\lambda)] - F_1(\lambda)\,D_\lambda\,[\phi_1 D_\lambda\,F_2(\lambda)] = 0 \tag{428}$$

or
$$D_\lambda\,[\phi_1 F_2(\lambda)\,D_\lambda\,F_1(\lambda) - \phi_1 F_1(\lambda)\,D_\lambda\,F_2(\lambda)] = 0 \tag{429}$$

giving
$$F_2(\lambda)\,F_1'(\lambda) - F_1(\lambda)\,F_2'(\lambda) = \frac{C}{\phi_1} \tag{430}$$

Now we may always define $F_1(\lambda)$ and $F_2(\lambda)$ so that $C = 1$, and under these conditions
$$A = \frac{F_2(\lambda)\,\phi_1(\lambda')\,[F_2(\lambda_3)\,F_1(\lambda') - F_1(\lambda_2)\,F_2(\lambda')]}{F_1(\lambda_1)\,F_2(\lambda_2) - F_1(\lambda_2)\,F_2(\lambda_1)}\left[-4\pi\int_{\lambda'-\varepsilon}^{\lambda'+\varepsilon}\frac{\sigma\,d\lambda}{h_1 h_2 h_3}\right] \tag{431}$$

and thus
$$v = \frac{[F_2(\lambda_1)\,F_1(\lambda) - F_1(\lambda_1)\,F_2(\lambda)]\,[F_2(\lambda_2)\,F_1(\lambda') - F_1(\lambda_2)\,F_2(\lambda')]}{F_1(\lambda_1)\,F_2(\lambda_2) - F_2(\lambda_1)\,F_1(\lambda_2)}$$

$$\times \left[-4\pi\phi_1(\lambda') \int_{\lambda'-\varepsilon}^{\lambda'+\varepsilon} \frac{\sigma d\lambda}{h_1 h_2 h_3} \right] (1-\Lambda)$$

$$+ \frac{[F_2(\lambda_2) F_1(\lambda) - F_1(\lambda_2) F_2(\lambda)][F_2(\lambda_1) F_1(\lambda') - F_1(\lambda_1) F_2(\lambda')]}{F_1(\lambda_1) F_2(\lambda_2) - F_2(\lambda_1) F_1(\lambda_2)}$$

$$\times \left[-4\pi\phi_1(\lambda') \int_{\lambda'-\varepsilon}^{\lambda'+\varepsilon} \frac{\sigma d\lambda}{h_1 h_2 h_3} \right] \Lambda. \tag{432}$$

The next step is to regard this as a differential equation in terms of μ. It is of infinite order but linear. It can be solved by a method similar to that used in the other examples by finding the roots of

$$F_1(\lambda_1) F_2(\lambda_2) - F_1(\lambda_2) F_2(\lambda_1) \tag{433}$$

regarded as a function of D_μ. The two terms on the right-hand side of (432) are then expanded in partial fractions in the usual manner and constitute an infinite number of second order equations for D_μ. In fact, the equations to be solved are identical with the original Laplace or Poisson equation, except that the operator D_λ is now replaced by a definite number determined by the roots of (433). Thus the solution of the original equation may be obtained by repeating the process outlined above with respect to one variable, a finite number of times. In actual practice it may be found that the solution of the second order equations which arise in this procedure can be reduced by simple transformations to equations whose solutions have already been worked out and tabulated. This will introduce slight modifications in the method, but these will have to be considered with reference to the special cases in which they arise. They should, however, present no serious difficulties.

THEOREM ON POTENTIAL DUE TO GREEN.

Green has shown that if the Green's function G for any closed surface is known, then the potential for any arbitrary distribution of potential V_1 over this surface can be calculated from the formula

$$V = -\iint V_1 \sigma dS \tag{434}$$

where

$$4\pi\sigma = -\frac{\partial G}{\partial n} \tag{435}$$

or σ is the surface density of the electricity induced on dS by a charge equal to unity at the point P. A proof of this theorem

will be found in Maxwell, "Electricity and Magnetism" (third edition, Vol. 1, §97*b*, p. 133).

ANALOGOUS THEOREM IN HEAT PROBLEMS.

A theorem similar to the above enables the solution for an initial arbitrary distribution of heat, and a given arbitrary temperature applied to the boundaries of the space, to be determined when the Green's function is known. The theorem is as follows (A proof of it may be found in Carslaw, "Mathematical Theory of Heat," second edition, pp. 170 and 171):

Let u be the Green's function for the space under consideration. The term "space" has already been used, and is defined by the Laplace equation and boundary conditions for the solid. Then the solution for

$$\left.\begin{array}{l} v = f(x,y,z) \text{ initially inside } S \\ v = \phi(x,y,z,t) \text{ at } S \text{ when } t > 0 \end{array}\right\} \tag{436}$$

is given by

$$v = \iiint uf(x,y,z)\,dx\,dy\,dz + \kappa \int_0^t d\tau \iint \phi(x,y,z,\tau) \frac{\partial u(t-\tau)}{\partial n}\,dS \tag{437}$$

where $u(t-\tau)$ is the value of the Green's function when t is replaced by $t-\tau$, and $\partial/\partial n$ denotes differentiation along the normal drawn into the space. The volume integral is taken throughout the volume of the solid and the surface integral over the surface of the solid.

It should now be evident that the classification which has already been given of Green's functions in heat problems into two classes, one with respect to an applied constant temperature and the other with respect to an instantaneous point source is not necessary. If we have the solution for an instantaneous point source, the other follows from the above theorem and even Duhamel's theorem may be dispensed with.

We shall now give two examples on the use of the above theorems, one with reference to potential problems and the other a heat problem.

THE GREEN'S FUNCTION IN POTENTIAL FOR THE SPACE BOUNDED BY $x=0$, $x=a$, $y=0$, $y=b$.

If a line charge is placed at $(x_1 y_1)$ there is no variation with respect to z and the equations to be solved are

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -4\pi\rho \tag{438}$$

where

$$v = 0 \qquad x = 0 \tag{439}$$
$$v = 0 \qquad x = a \tag{440}$$
$$v = 0 \qquad y = 0 \tag{441}$$
$$v = 0 \qquad y = b \tag{442}$$

Integrating with respect to x in the usual manner

$$v = \left[\frac{\sin \alpha x \sin \alpha (a - x_1)}{\alpha \sin \alpha a}(1 - X) + \frac{\sin \alpha (a - x) \sin \alpha x_1}{\alpha \sin \alpha a} X\right] \int_{x_1 - \varepsilon}^{x_1 + \varepsilon} 4\pi\rho dx \tag{443}$$

where X is the Heaviside unit function with respect to x and $\alpha = \frac{d}{dy}$.

Expanding the first term on the right-hand side in partial fractions

$$\frac{\sin \alpha x \sin \alpha (a - x_1)}{\alpha \sin \alpha a} \int_{x_1 - \varepsilon}^{x_1 + \varepsilon} 4\pi\rho dx$$
$$\longleftrightarrow \left[\frac{x(a - x_1)}{a} + \frac{2}{a} \Sigma (-1)^n \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi (a - x_1)}{a}}{\alpha^2 - \frac{n^2\pi^2}{a^2}}\right] \int_{x_1 - \varepsilon}^{x_1 + \varepsilon} 4\pi\rho dx \tag{444}$$

The first term gives zero, since when $x \neq x_1$, the value of the integral is zero. The value of the other terms is obtained by finding functions $A_1, A_2, \ldots A_n \ldots$ such that

$$A_n = \frac{1}{\alpha^2 - \frac{n^2\pi^2}{a^2}} \int_{x_1 - \varepsilon}^{x_1 + \varepsilon} 4\pi\rho dx \tag{445}$$

or

$$\left(\frac{d^2}{dy^2} - \frac{n^2\pi^2}{a^2}\right) A_n = \int_{x_1 - \varepsilon}^{x_1 + \varepsilon} 4\pi\rho dx \tag{446}$$

Since the value of the integral is zero everywhere except when $y = y_1$, we can write in the usual way

$$A_n = A \sinh \frac{n\pi y}{a}(1 - Y) + A \frac{\sinh \frac{n\pi (b - y)}{a} \sinh \frac{n\pi y_1}{a}}{\sinh \frac{n\pi (b - y_1)}{a}} Y \tag{447}$$

where the value of A is determined by substituting in the differential equation (446) and getting, when $y = y_1$,

$$\left[-\frac{n\pi}{a}\cosh\frac{n\pi y_1}{a} - \frac{n\pi}{a}\frac{\cosh\frac{n\pi}{a}(b-y_1)\sinh\frac{n\pi y_1}{a}}{\sinh\frac{n\pi(b-y_1)}{a}} \right] A = \int_{y_1-\varepsilon}^{y_1+\varepsilon}\int_{x_1-\varepsilon}^{x_1+\varepsilon} 4\pi\rho dxdy \tag{448}$$

or

$$A = \frac{4\pi Q}{\frac{n\pi}{a}}\frac{\sinh\frac{n\pi(b-y_1)}{a}}{\sinh\frac{n\pi b}{a}}. \tag{449}$$

Thus the Green's function for this space becomes

$$v \longleftrightarrow \frac{8\pi Q}{a}\sum_{n=1}^{n=\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x_1}{a}}{\frac{n\pi}{a}}\left[\frac{\sinh\frac{n\pi y}{a}\sinh\frac{n\pi(b-y_1)}{a}}{\sinh\frac{n\pi b}{a}}(1-Y) + \frac{\sinh\frac{n\pi y_1}{a}\sinh\frac{n\pi(b-y)}{a}}{\sinh\frac{n\pi b}{a}}Y\right] \tag{450}$$

POTENTIAL OF THE SPACE BOUNDED BY x = 0, x = a, y = 0, y = b, WHEN ONE OF THE SIDES IS RAISED TO AN ARBITRARY POTENTIAL, AND THE OTHER SIDES ARE KEPT AT ZERO POTENTIAL.

Let

$$\left.\begin{array}{ll} v = 0 & x = 0 \\ v = 0 & x = a \\ v = f(x) & y = 0 \\ v = 0 & y = b \end{array}\right\} \tag{451}$$

Using the results of the last example

$$4\pi(\sigma)_{y=0} = -\left(\frac{\partial G}{\partial y}\right)_{y=0} \longleftrightarrow \frac{8\pi Q}{a}\sum_{n=1}^{n=\infty}\sin\frac{n\pi x}{a}\sin\frac{n\pi x_1}{a}\frac{\sinh\frac{n\pi(b-y_1)}{a}}{\sinh\frac{n\pi b}{a}} \tag{452}$$

Note: The first term in the Green's function in (450) is the only one used since $0 < y_1$. Hence, using the theorem due to Green (434),

$$v \longleftrightarrow \frac{2}{a}\sum_{1}^{\infty}\frac{\sinh\frac{n\pi(b-y_1)}{a}}{\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x_1}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx. \tag{453}$$

This result may easily be extended to the case where the four sides

are kept at arbitrary potentials and the result checked with those given by Byerly, "Fourier Series," Art. 59, p. 103.

When $a \to \infty$ and $b \to \infty$, (453) may be written

$$v \longleftrightarrow \lim_{a\to\infty} \lim_{b\to\infty} \frac{2}{a} \sum_{1}^{\infty} \frac{e^{\frac{n\pi(b-y_1)}{a}} - e^{-\frac{n\pi(b-y_1)}{a}}}{e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}} \times \int_0^a \sin\frac{n\pi x_1}{a} \sin\frac{n\pi x}{a} f(x)\, dx$$

$$\longleftrightarrow \lim_{a\to\infty} \frac{2}{\pi} \sum_{1}^{\infty} \frac{\pi}{a} e^{-\frac{n\pi y_1}{a}} \int_0^a \sin\frac{n\pi x_1}{a} \sin\frac{n\pi x}{a} f(x)\, dx$$

$$\longleftrightarrow \frac{2}{\pi} \int_0^\infty d\alpha \int_0^\infty e^{-\alpha y_1} f(x) \sin\alpha x_1 \sin\alpha x\, dx \qquad (454)$$

where x_1 and y_1 are the coördinates of the point. This may be checked with the result given by Byerly, "Fourier Series," p. 77.

FLOW OF HEAT IN A BAR OF LENGTH l, ONE END OF WHICH IS SUDDENLY RAISED TO A TEMPERATURE v_1 AT t = 0, AND THE OTHER END IS KEPT AT ZERO TEMPERATURE.

The space for this Green's function is given by

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\kappa} \frac{\partial v}{\partial t} \qquad (455)$$

$$v = 0 \qquad x = 0 \qquad (456)$$

$$v = 0 \qquad x = l \qquad (457)$$

This Green's function has already been determined in (332) and is equal to

$$u \longleftrightarrow \frac{2Q\kappa}{lc} \sum_{1}^{\infty} \sin\frac{n\pi x}{l} \sin\frac{n\pi x_1}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} 1 \qquad (458)$$

when $x = l$

$$\frac{\partial u}{\partial n} = \frac{du}{dx} \longleftrightarrow \frac{2Q\kappa}{lc} \sum_{1}^{\infty} (-1)^n \frac{n\pi}{l} \sin\frac{n\pi x_1}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} 1 \qquad (459)$$

Hence the temperature at any point of the rod when the end $x = l$ is suddenly raised to a temperature v_1 at $t = 0$ is

$$v = \int_0^t v_1 \frac{du(t-\tau)}{dx}\, d\tau$$

$$\longleftrightarrow \frac{2Q\kappa v_1}{lc} \left[\Sigma (-1)^n \frac{\sin\frac{n\pi x_1}{l}}{\frac{n\pi}{l}} e^{-\kappa \frac{n^2\pi^2}{l^2}(t-\tau)} \right]_{\tau=0}^{\tau=t}$$

$$\longleftrightarrow \frac{2Q\kappa v_1}{lc}\left[\frac{l}{\pi}\Sigma(-1)^n\frac{\sin\frac{n\pi x_1}{l}}{n}-\frac{l}{\pi}\Sigma(-1)^n\frac{\sin\frac{n\pi x_1}{l}}{n}e^{-\kappa\frac{n^2\pi^2}{l^2}t}\right]$$

$$\longleftrightarrow \frac{Q\kappa v_1}{c}\left[\frac{x_1}{l}+\frac{2}{\pi}\Sigma(-1)^n\frac{\sin\frac{n\pi x_1}{l}}{n}e^{-\kappa\frac{n^2\pi^2}{l^2}t}\right] \qquad (460)$$

CONCLUSION.

It is clear from the above examples that the solution for the Green's function in the general case of orthogonal curvilinear coördinates can be obtained in an automatic manner. In all the spaces considered, finite boundaries for each of the coördinates have been used. The cases where one or more of the boundaries are at infinity or have other peculiarities may be derived from the solutions for the finite case by means of a limiting process. In general the above method will not give results when one or more of the boundaries are at infinity or have other peculiar characteristics, due to the fact that in such cases some process similar to fractional differentiation generally arises and it is not proposed to attempt to discuss this here. The application of the limiting process to the solution for finite boundaries does not, however, present any serious difficulties. It has been illustrated in a comparatively simple case.

When the manner in which the various conditions are inserted is carefully examined, it would appear fair to say that the solution obtained by this method, regarded as an H-function, converges to the solution of the original set of equations. The only concern, from the viewpoint of the theory of H-functions, need be to examine whether it is an H-uniformly convergent solution or not. If it is H-uniformly convergent it may be used directly for obtaining numerical results. If it is not H-uniformly convergent at some or all points in the range, it requires further examination to determine which of its properties represent a solution of the original equation and which do not. Just as in the solution of problems by means of algebra, letting x equal the unknown, we sometimes obtain more than one value of x, one of which satisfies the original problem and the others manifestly do not, so in the solutions obtained by this method, values are sometimes obtained which are not applicable to the problem in hand. For instance, consider the Green's function for a rod of length l with both ends maintained at zero temperature,

$$G(x,y,t) \longleftrightarrow \frac{2}{l} \Sigma \sin \frac{n\pi x}{l} \sin \frac{n\pi y}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t} \tag{461}$$

when $t = 0$

$$G(x,y,0) \longleftrightarrow \frac{2}{l} \Sigma \sin \frac{n\pi x}{l} \sin \frac{n\pi y}{l} \tag{462}$$

This H-function, as has already been pointed out, consists of an infinite number of arches in the interval $0 \leq x \leq l$. The arches are bounded by definite curves on the upper and lower limits. When $x = y$ the maximum value of $G(x, y, 0)$ is infinity. We might thus say $G(x, y, 0)$ has an infinite number of points in common with the H-function

$$\left.\begin{array}{ll} z = 0 & x < y \\ z = 0 & x > y \\ 0 \leq z \leq 1^{\infty 1} & x = y \end{array}\right\} \tag{463}$$

but it has an infinity of other values in addition. But (463) is similar to the customary initial conditions used in determining the Green's function, hence the Green's function as thus determined satisfies these values initially but has also other values in addition. In cases, however, where the initial distribution of heat is over a finite area, so that the temperature everywhere is finite, the series derived are H-uniformly convergent almost everywhere, and can be used for calculating numerical values at all points where they do converge H-uniformly. In these cases the proper values at the non-H-uniformly convergent points must be picked out and the others rejected.

From the facility with which the method can be applied to Laplace's equation in orthogonal curvilinear coördinates it should be obvious that using it and the theorems on the use of Green's functions given above, practically all the problems in a standard work such as Carslaw, "Mathematical Theory of Heat Conduction" (second edition, 1921), or Byerly, "Fourier Series and Spherical Harmonics," can be readily solved. The method should also be applicable to other branches of mathematical physics when the equations involved are put in a suitable form such as was necessary in the modification of the equation of conduction of heat in (298), and should also be of assistance in obtaining solutions of problems which have not been already worked out. No attempt has been made here to give the solution of new problems on account of the labor involved in checking the results.

It is to be hoped that the generality of the method will induce others to work upon the theory of the principles involved along such lines as have been outlined. The attempt here has been to point out some of the fundamental ideas which seem to underly the work of Heaviside and to show wherein they differ from the ordinary mathematical conceptions. It has, however, been impossible to go into great detail. It would also appear from the results obtained that the determination of the Green's function is of fundamental importance in all such problems and that a collection of these functions, for various types of spaces, in easily accessible form, would be of considerable practical value.

The point of view that the use of ε does not prove equality as in ordinary mathematics, but only shows the degree of the convergence of one H-function to another, leads to interesting results in connection with the interpretation of generalized series, divergent series, fractional differentiation, etc., used by Heaviside. However, the results obtained in this connection are not very complete and cannot be gone into here on account of the already great length of this paper.

In conclusion I wish to thank Dr. E. J. Berg for the many discussions on the points involved and for his never-failing interest during the course of the work.

Prevention of Frost in Orchards. (Clip Sheet, U. S. Department of Agriculture, No. 387, November 23, 1925.)—A "windjammer" is the local name in California fruit-growing districts for a machine devised to raise the temperature of an orchard by the process of mixing the air during critical periods when the fruit is threatened by frost. Since the establishment of its fruit-frost service in 1917, the Weather Bureau has had occasion to make practical tests with a number of these devices, but up to the present time has found no "windjammer" which can be recommended for complete protection from low temperatures. The use of many small orchard heaters is to-day the only practical means of raising the temperature of the air near the ground sufficiently to protect the fruit on the trees.

NOTES FROM THE RESEARCH LABORATORY, EASTMAN KODAK COMPANY.*

PLASTICITY IN RELATION TO GELATIN.[1]

By S. E. Sheppard.

GELATIN solutions above a certain temperature behave as viscous solutions, not as plastic systems. This temperature is not independent of concentration, but depends on concentration, on the mode of preparation (dispersion), on the p_H and other factors. Gelatin jellies, below the setting point, are highly elastic, but show pseudo-plastic flow when exposed for long periods to loads above the elastic limit.

The Motion of an Air-bubble Rising in Water. OTOGORO MIYAGI, Japan. (*Phil. Mag.*, July, 1925.)—This paper by the professor of hydraulics and hydraulic engineering in the Tohoku University, Sendai, is at once interesting and important. Its English is so excellent that one must watch with care in order to catch the elusive trespass of our idiom, which just suffices to show that the author is not using his native language.

On this subject very little work has been done. H. S. Allen a quarter of a century ago studied the progress of small bubbles, not more than .4 mm. in radius. This Japanese investigation extends the range to radii of more than 3 mm. There still remain those large unstable bubbles of large radius.

The bubbles were released at the bottom of a glass tank. As they moved up through the water they were illuminated at intervals by the light from a 500-candlepower incandescent lamp, that made its way through the slots of a rotating disc. This light traversed about an inch of water and projected the bubble therein upon a rotating cylinder of bromide paper. At the same time this received another beam of light from a vibrating tuning-fork. By every flash of light let pass by the disc an image of the bubble and of the nozzle from which it had started was recorded. Thus at intervals the position of the bubble was made known, while the tuning-fork trace provided the time record. The bubbles were released slowly from the nozzle, so that the diameter could be measured by microscope just before they started upward. Successive bubbles were alike in all respects.

The photographs show the bubble in about forty stages in its

* Communicated by the Director.

[1] Communication No. 227 from the Research Laboratories of the Eastman Kodak Company, and published in *J. Phys. Chem.*, **29**, Oct., 1925, p. 1224.

path up through the column of water 20 cm. high. Its changes of form as well as its position are seen. From the distance it is above the nozzle its velocity can be followed. It was found that for three or four centimetres after its release the bubble gained in velocity until a maximum was reached at about which velocity it continues to progress upward to the top. The magnitude of this maximum is a function of the radius of the bubble. If the radii of the bubbles be plotted as abscissas and the velocities as ordinates, the curve is practically a straight line for radii up to 1.2 mm. The velocity of a bubble .25 mm. in radius is 9.85 cm./sec., and that for a radius 1.2 mm. is 24.5 cm./sec. For bubbles with radii larger than 1.2 mm. the curve bends and finally starts downward. This change in the course of the curve means that an air-bubble of radius 1.65 mm. has a greater velocity than any other, and that its velocity and consequently the highest velocity attained by any air-bubble in water, is 27.8 cm./sec. For larger bubbles the velocity falls until is becomes constant at 23.1 cm./sec. These results hold for water at 18° C., the temperature making a slight difference by reason of its influence on viscosity.

On the successive photographs of the bubble let a series of vertical lines be drawn through the centre of each image of the nozzle. If the bubble rose vertically, each image of it would be vertically above the nozzle. This is not the case. The image is sometimes to the right and sometimes to the left of the vertical line. "It loiters about that line, dashing in every direction, tracing a course like a helix round it, as observed from the top of the experiment vessel looking down into the water." The reason for the deviation of the path from the vertical line is the lack of symmetry in the form of the bubble, and this form changes from moment to moment, as the photographs show. In a rough way the bubble is ellipsoidal in shape. Its major axis, then, is "always perpendicular to its course, that is, it moves always with its flattened face directed ahead."

A theoretical discussion follows. Sir G. G. Stokes and Allen have each given a formula for calculating the velocity of the bubble, when its radius is known. Stokes' formula gives values of the velocities that are enormously larger than the observed values, except for small bubbles. Allen's formula fits the facts better, but leaves much to be desired. The author derives a fresh formula, that gives velocities agreeing with those observed within a few per cent., instead of within a few hundred or thousand per cent., as is the case with the two other formulas.

The resistance offered to the motion of the bubble was found to be in proportion to the cube of its radius, or to its volume. It was surmised that the smallness of the water column might affect the terminal velocities attained. A larger vessel was, therefore, used and velocities 4 per cent. larger than the previous ones were obtained.

G. F. S.

NOTES FROM THE U. S. BUREAU OF CHEMISTRY.*

EFFECT OF DEHYDRATION UPON BACTERIAL FLORA OF EGGS.[1]

By George G. De Bord.

[ABSTRACT.]

THE count of viable bacteria in freshly prepared dehydrated egg varies, in general, with the quality of the raw product and the method of dehydration. The counts in the product prepared from whole eggs by the spray process varied from 350 in the good egg to 1,160,000 in the spots. In the product prepared by the vacuum-drum process from whole egg the counts varied from 45,000 in the good egg to 2,400,000 in the rots. In general, the yolk showed a higher number than the whites from the same whole egg.

The plate count of spray-process dehydrated egg held in storage depends on the initial count, the length of time in storage, and the temperature of storage. An initial count of 350 in good egg decreased to 300 in ten months when held at 37° C. and at 20°, while a count of 235,000 in rotten eggs decreased to 1350 when held at 37°. In one sample of the rotten eggs held at 20° the count increased from 235,000 to 430,000 in three months. A second sample showed a decrease from 250,000 to 95,000 at the end of three months, and a further decrease to 41,000 at the end of ten months.

The odor which is characteristic of poor-quality eggs is lost to some extent during the dehydration process.

An odor similar to rancidity develops in egg powder held at the various temperatures. This odor was most pronounced in rotten eggs held at 37° C.

A DEVICE FOR PREPARING LIGHT LEATHER SAMPLES FOR ANALYSIS.[2]

By R. W. Frey.

[ABSTRACT.]

THE machines for preparing heavy leathers for analysis are not suitable for light leathers, and the use of shears or a knife for such work is a tedious process. A device, consisting essentially of

* Communicated by the Chief of the Bureau.

[1] Published in *Jour. Agri. Research*, 31, No. 2 (July 15, 1925), pp. 155–164.

[2] Published in *J. Am. Leather Chemists Asso.*, 20 (Oct., 1925): 470.

a set of safety razor blades, fastened to the plunger of an eyelet press, suitably mounted and secured to a machined cast-iron block, has been designed in the Bureau of Chemistry to prepare bookbinding, shoe upper, glove, and similar leathers. It has proved effective and simple of operation.

THE DISTRIBUTION OF TANNIN IN THE AMERICAN CHESTNUT TREE WITH PARTICULAR REFERENCE TO STUMPS AND ROOTS.[1]

By R. W. Frey and L. R. Leinbach.

[ABSTRACT.]

DATA have been compiled on the tannin content of the American chestnut tree, with particular reference to stumps and roots, and to the horizontal distribution of tannin. Averages of the tannin content of different parts of the tree are as follows:

Location.	Per Cent. Tannin.
Stump bark, up to 5 feet	12.7
Trunk heartwood, centre	8.6
Trunk heartwood, edge	15.3
Stump heartwood, centre	9.3
Stump heartwood, intermediate	13.4
Stump heartwood, edge	16.4
Rootwood	17.4
Rootbark	31.4

Stumpwood, rootwood, and rootbark are thus shown to have a comparatively high tannin content, and the possibility of utilizing them as a commercial source of tannin is suggested. On the basis of an average for commercial chestnut wood of from 7.5 to 8.5 per cent. tannin, the results on the stumps and roots indicate a yield from them approximately twice as great as that from the commercial run of wood. Because of the normally excellent and rapid development of second growth from chestnut stumps, it might seem unwise to advocate their removal, were it not for the very rapid advance of the chestnut blight. Results on the horizontal distribution of tannin suggest a further study of the utilization of slabwood, since it is clearly shown that the tannin content of the edge of the heartwood is much higher than that of the centre.

One set of results on a chestnut oak tree indicates a relationship between the tannin content of the stumpwood, rootwood, and rootbark similar to that found for the chestnut strees.

[1] Published in *J. Am. Leather Chemists Asso.*, 20 (Oct., 1925): 457.

THE CHEMICAL COMPOSITION OF THE BITONTO TYPE OF ITALIAN OLIVE OIL.[4]

By Walter F. Baughman and George S. Jamieson.

[ABSTRACT.]

THE chemical composition of Italian olive oil of the Bitonto type, as compared with that of California olive oil expressed from Mission olives, is as follows:

		Italian Olive Oil (Bitonto Type).	California Olive Oil from Mission Olives.
Glycerides of	Oleic	83.1	84.4
	Linolic	3.9	4.6
	Myristic	trace	trace
	Palmitic	9.2	6.9
	Stearic	2.0	2.3
	Arachidic	0.2	0.1
Unsaponifiable matter		1.1	1.0

The Italian oil contains 2.1 per cent. more of the saturated acid glycerides than the California oil. The widest difference is in the percentages of palmitic acid glycerides. The California oil contains 2 per cent. more of the unsaturated acid glycerides.

THE THEORY OF THE SCHROTEFFEKT.

BY

THORNTON C. FRY, Ph.D.

A CORRECTION.

IN THE concluding pages of my paper on the "Theory of the Schroteffekt" (this JOURNAL, Vol. 199, p. 203, 1925), I applied the general methods which I had derived to the solution of a particular example. This example, which corresponds very closely to the conditions under which the Schroteffekt is investigated in the laboratory, had previously been considered by several other investigators. In comparing the various results, I remarked that my solution was

> "identical with the result obtained by Schottky, Ornstein and Burger, and Fürth, except for the factor $\left[1 + \frac{R'^2\kappa}{L}\right]$, which was not included in their results. . . . As this factor generally differs from unity by only a small amount, their results are usually sufficiently accurate to satisfy laboratory requirements."

[4] Published in *J. Oil and Fat Ind.*, 2 (Oct., 1925): 110.

As regards Professor Schottky, this statement is incorrect, for one of his papers contains a footnote in which the complete solution is derived (*Annalen der Physik*, Vol. 57, p. 541, 1918).

The Radium Industry. C. MATIGNON, Professor in the College de France. (*Revue Scientifique*, Aug. 8, 1925.)—It is not yet twenty-eight years since Mme. Curie announced to the Academy of Sciences that both pitchblende and chalcolite possess radio-active properties more intense than the uranium contained in them. Then there followed the establishment of the existence of two new elements, polonium, chemically akin to bismuth, and radium, allied to barium. At that time the pitchblende of Joachimsthal in Bohemia provided uranium salts for the world, and it was from the residue left after extraction that the new element, radium, was derived. The Austrian Government owned the mine, and ceased the exportation of the mineral in an endeavor to set up a monopoly of radium. Along with this attempt the well-known Radium Institute at Vienna was created. Quite naturally other sources of uranium were sought throughout the world. To the deposits listed in Cornwall, Portugal, and Saxony were added those found in Mexico and in the United States of America, but more important than these was the carnotite of Colorado and Utah. At first the mineral was exported, as in 1912, when 1200 tons were sent to France to have the radium extracted, but from that time onward the American plants got into operation, so that they produced 10.5 grams of radium in 1913, and 22.4 grams the following year. The exportation of the mineral continued, however, to the beginning of the war. Though it was necessary to treat 400 tons of carnotite to get 1 gram of radium, the process was relatively simple, and the United States became the chief source of radium. This condition was changed when the rich veins of pitchblende discovered in the Belgian Congo in 1913 began to furnish radium to the market in 1922 or 1923. A factory at Oolen, near Antwerp, even in the latter year was producing 4 grams per month. The African mineral is about thirty times as rich in uranium oxide as is carnotite.

It is estimated that in the entire world there are about 300 grams of radium, worth 300,000,000 francs. The United States has in its possession half of this total quantity, of which 120 grams are in its hospitals and other medical institutions.

The following shows the price in dollars of one milligram of radium in different years:

1904.....	$10 to $25	1909–1910....	$75 to $135
1905.....	25 to 50	1911–1912....	$150
1906.....	60	1912–1914....	$180
1915.....	$160		
1916–1922.	$120, $110, $105		
1923.....	70		

The advent of radium from American carnotite reduced the price from $160 to about $110. Belgian production further lowered the cost to $70. G. F. S.

THE FRANKLIN INSTITUTE.

(Proceedings of the Stated Meeting, Wednesday, November 18, 1925.)

THE stated meeting of The Franklin Institute was called to order by the President, Doctor Eglin, at eight-sixteen P.M.

The President called for the reading of the minutes of the last meeting. The Secretary announced that the minutes of the last meeting had been published in full in the November number of the JOURNAL of the Institute. He therefore moved that the minutes be approved as printed. This motion was adopted unanimously.

The Secretary then gave notice that at the next stated meeting of the Institute, to be held in December, action would have to be taken upon the proposed amendments to the By-laws which were recommended by the Board of Managers at the stated meeting in November.

There was no other business.

The President presented as the lecturer of the evening Prof. Ernest W. Brown, Professor of Mathematics at Yale University, who lectured upon "The Recent Solar Eclipse and Its Results." Professor Brown's lecture was heard with absorbed interest by those present.

The meeting adjourned at nine-twenty-one P.M., with a rising vote of thanks to the speaker.

Respectfully submitted,

HOWARD MCCLENAHAN,
Secretary.

November 19, 1925.

COMMITTEE ON SCIENCE AND THE ARTS.

(Abstract of Proceedings of Stated Meeting held Wednesday, November 4, 1925.)

HALL OF THE COMMITTEE,
PHILADELPHIA, November 4, 1925.

MR. HAROLD CALVERT *in the Chair.*

The following reports were presented for final action:

No. 2821: Gray Sheet Metal Cutter.

The specific purpose of this invention is to provide a machine for cutting metal sheets and plates into irregular shapes, leaving them in a finished condition.

The operating part of the machine consists of a reciprocating cutter having a rectangular cross-section. The stock is fed under the cutter between feed rollers, the two lower rollers, which are made of case-hardened steel, form the edges of the die into which the cutter passes. These rollers are separated by a spacing washer and the end of a steel bar placed in the channel so formed serves as the third side of the die in which the cutter operates.

The stock is automatically fed forward at every upward stroke of the cutter and is held in place between the rollers at every downward stroke.

United States Patents Numbers 1,098,376 and 1,098,377 were granted for this device on June 2, 1914, and on the recommendation of the Committee the Certificate of Merit was awarded to Mr. Charles B. Gray, of Buffalo, N. Y., for his invention of the Gray Sheet Metal Cutter.

No. 2830: Fynn-Weichsel Motor.

"This machine is a synchronous-induction motor, either two- or three-phase, having an automatic self-generated direct current to obtain the synchronous speed but with characteristics quite different from the ordinary synchronous motor. It may be used to produce mechanical power at unity power factor, or it may produce power and also raise the power factor of other motors in the same installation.

"In appearance it resembles an ordinary slip-ring motor. In an ordinary induction motor the current is supplied to the stationary winding and the rotor is either a permanently short-circuited squirrel-cage winding or is connected to resistances which are used for starting, after which the rotor is short-circuited. In the Fynn-Weichsel motor the line current is led into the rotor through the slip-rings. The stationary part, or stator, is also of common construction, being similar to that of an ordinary two-phase motor. Both phase windings are used in starting, one being short-circuited thereafter and carrying no current at synchronous speed while the other is supplied with direct current from the rotor after reaching full speed.

"The essential difference lies in the revolving part or rotor. This also has two windings, one of which is connected to slip-rings. Alternating current, two- or three-phase, is supplied from the line through brushes resting on the rings and is thus conveyed to the winding. This causes the rotor to start as an ordinary induction motor, to accelerate to nearly synchronous speed and after the resistances in the stator windings are short-circuited to fall into synchronism. The second winding is also connected to a commutator. The brushes resting thereon carry the direct current which is generated to the above-mentioned stationary, or field winding.

"The motor is supplied with a starting device so that when the line current is supplied through a switch to the slip-rings, resistance is cut out of the starting windings on the stator frame and the motor starts and reaches about 96 per cent. of synchronous speed just as a slip-ring motor will. When the motor is running at less than synchronous speed, there is generated in this winding an alternating current whose frequency is low and proportional to the slip, which is the difference between the actual and synchronous speeds. Consequently, as the motor speed increases, the slip and frequency of this current decrease and the current approaches more and more to the characteristics of direct current. Finally, a condition exists, when the field is supplied with direct current, and the motor then operates at synchronous speed.

"When the load reaches a value of about 150 per cent. instead of dropping out of step and stopping, as would the ordinary motor, the Fynn-Weichsel simply lessens its speed and continues running as an induction motor. Should this load be reduced the motor will again return to synchronous speed and

such operations would continue indefinitely. It is not until the load is increased to approximately three times full load that the motor will stop entirely."

A number of United States Patents have been granted on this motor and on the recommendation of the Committee, the John Price Wetherill Medal was awarded to the Wagner Electric Company of Saint Louis, Mo., with special mention of Val A. Fynn and H. Weichsel, "in consideration of the progress in the electric art which this invention marks and of the development of a motor of excellent design and construction."

The following report was presented for first reading:

No. 2825: Hilger Interferometer.

GEO. A. HOADLEY,
Secretary to Committee.

BARTOL RESEARCH FOUNDATION.

APPOINTMENT OF FIRST FELLOW.

DR. ARTHUR BRAMLEY has been appointed Fellow of the Bartol Research Foundation of The Franklin Institute, and has taken up his duties.

The laboratories of the Foundation have been equipped with all primary essentials for investigations in modern physics, particularly those pertaining to fundamental problems in electricity. From time to time the Foundation will appoint Fellows who will pursue investigations in its laboratories. Applications for Fellowships should be made to the Chairman of the Personnel Committee, 127 North Nineteenth Street, Philadelphia, from whom further particulars may be obtained.

THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, MEETING AT SOUTHAMPTON.

(Extract from *Engineering*, London, September 25, 1925.)

"*Attempt to Detect a Corpuscular Radiation of Cosmic Origin.* Prof. W. F. G. Swann, now of Yale College, New Haven, discussed in this paper another fundamental problem. We do not know, as Doctor Simpson subsequently also emphasized, how the loss of the earth's charge through the atmosphere is being maintained. If cosmic electrons were to effect that replenishment, some 1500 should enter the earth per square centimetre per second. Those electrons should produce ions near the earth in numbers thousands of times greater than those observed and otherwise accounted for.

"Professor Swann, whose experiments are being continued under the auspices of The Franklin Institute, connected two hollow copper cylinders to a quadrant of an electrometer; each cylinder was jacketed by a larger copper cylinder, the two latter being joined to the ends of a megohm resistance connected at the midpoint with the electrometer case. The battery (150 volts) connections were such that the ionising currents in the two systems balanced. Each of the inner cylinders could be filled with solid copper. From the observations it was possible to estimate the rate of charging the copper by an absorption of radiation of corpuscular origin, such as might penetrate into our atmosphere from outside. The results, as in previous experiments at Kew, showed that the corpuscular current absorbed by the cylinder was certainly

less than 1 per cent. of that which would have been expected on the basis of an explanation of the earth's charge by such corpuscular currents."

The investigation by Professor Swann referred to above was carried out under the auspices of the Bartol Research Foundation and with apparatus made in the machine shops of the Foundation.

MEMBERSHIP NOTES.

ELECTIONS TO MEMBERSHIP.

(Stated Meeting, Board of Managers, November 18, 1925.)

RESIDENT MEMBERS.

MR. JOSEPH P. BELL, Mechanical Engineer, 230 North Monroe Street, Media, Penna.

MR. ARCHIBALD C. EGLIN, JR., Apprentice in Transportation Department, Pennsylvania Railroad System, Merbrook Lane, Merion, Penna., and for mail, 1230 Eighteenth Avenue, Altoona, Penna.

MR. JOHN M. MCCLENAHAN, Montgomery Avenue, Villa Nova, Penna.

MR. NOEL T. WELLMAN, Superintendent, Delaware Works, General Chemical Company, 102 Poplar Street, Ridley Park, Penna.

NON-RESIDENT MEMBERS.

DR. ALICE G. BRYANT, Physician, 502 Beacon Street, Boston, Mass.

MR. WARD F. DAVIDSON, Electrical Engineer, 380 Pearl Street, Brooklyn, N. Y.

MR. OTTO GOTTSCHALK, Mechanical and Civil Engineer, Casilla Correo 1723, Buenos Aires, Argentina.

DR. LARS OLAI GRONDAHL, Director of Research, Union Switch and Signal Company, Swissvale, Penna.

PROF. S. S. KISTLER, Department of Chemistry, College of the Pacific, Stockton, Calif.

CHANGES OF ADDRESS.

MR. MERRITT H. ADAMS, Woodlawn Terrace, Beechhurst, L. I., N. Y.

MR. ROBERT ATKINSON, Hampton Institute, Va.

DR. LOUIS W. AUSTIN, Bureau of Standards, Washington, D. C.

MR. CHARLES M. BIDDLE, JR., Riverton, N. J.

MR. J. E. EGLESON, 734 Ocean Avenue, Brooklyn, N. Y.

LIEUT. A. H. GFRORER, 507 East Pearl Street, Cincinnati, Ohio.

MR. RICHARD P. HARVEY, 1212 Hilldale Avenue, Sherman, Los Angeles County, Calif.

MR. JOHN W. HORNSEY, Oakes-in-Norton, by Sheffield, England.

MR. PHILIP C. JONES, 514 Barwell Street, Akron, Ohio.

MR. J. L. KILPATRICK, Fulton Towers, 78 Harrison Street, East Orange, N. J.

MR. F. B. LYSLE, 906 Colonial Trust Building, Philadelphia, Penna.

MR. J. HARTLEY MERRICK, Hotel Pennsylvania, 3900 Chestnut Street, Philadelphia, Penna.

MR. J. VAUGHAN MERRICK, 253 South Sixteenth Street, Philadelphia, Penna.

MR. R. P. C. SANDERSON, 500 North Broad Street, Philadelphia, Penna.
MR. E. A. SMITH, 1281 Pat. Plk Road, Secaucus, N. J.
MR. S. D. SPRONG, Room 309, 25 Church Street, New York City, N. Y.
MR. WILLIAM G. WARDEN, 809 Witherspoon Building, Philadelphia, Penna.
MR. RAYMOND WHITE, 322 Condley Drive, Toledo, Ohio.
MR. JOHN S. WILSON, 122 South Eutaw Street, Baltimore, Md.
MR. G. H. WOODROFFE, in care of the Reading Iron Company, Reading, Penna.
DR. WINTHROP R. WRIGHT, Cavendish Laboratory, Cambridge University, Cambridge, England.

NECROLOGY.

Professor Charles F. Chandler, Emeritus Professor of Chemistry in Columbia University, died at Hartford, Conn., on August 25th after a short illness.

Doctor Chandler was born at Lancaster, Mass., on December 6, 1836, was educated in the Lawrence Scientific School of Harvard College and afterward at Göttingen and Berlin. He returned to America when nineteen years of age and became assistant in chemistry at Union College and soon afterward professor, a position which he held until 1864 when he went to Columbia University. He was one of the founders of the School of Mines in which he was professor of chemistry until his retirement as emeritus professor in 1911. He was adjunct professor of chemistry and medical jurisprudence at the College of Physicians and Surgeons, New York, from 1872 to 1876 and professor from 1876 to 1897.

As president of the Health Department of New York City, which office he held for ten years, he was active in many needed reforms, particularly in cleaning up unsanitary markets at a time when the city was threatened with an epidemic of typhoid fever.

His contributions to the sulphuric acid and petroleum industries were of the utmost importance and of permanent value. As a teacher of chemistry he was without a peer. He knew how to make the subject absorbingly interesting; by arousing the curiosity of his students he had no difficulty in holding their rapt attention throughout his lectures.

Doctor Chandler was recognized as Dean of American Chemists and received many academic and honorary degrees, among them being Ph.D. from Göttingen in 1856, honorary M.D. from New York University in 1873, LL.D. from Union in 1873, honorary Sc.D. from Oxford in 1900 and LL.D. from Columbia in 1911.

He was an honorary member of the leading scientific societies of the world. His connection with The Franklin Institute began in February, 1900, when he became a non-resident member. In 1911 he was made an honorary member "in recognitoin of the services which he has rendered to the cause of scientific education and the application of chemical theory in the advancement of the arts and industries."

Dr. Edward Goldsmith, 658 North Tenth Street, Philadelphia, Penna.

LIBRARY NOTES.

RECENT ADDITIONS.

ANDERSON, WILLIAM BALLANTYNE.—Physics for Technical Students in Colleges and Universities. Second edition revised and enlarged. 1925.

Association of Official Agricultural Chemists.—Official and Tentative Methods of Analysis. Second edition revised to July 1, 1924. 1925.

BLOCH, EUGÈNE.—The Kinetic Theory of Gases. No date.

BÖSEKEN, J.—Configuration of the Saccharids. No date.

CAMPBELL, CHARLES A. R.—Bats, Mosquitoes and Dollars. 1925.

DICK, OTTO.—Die Feile und ihre Entwicklungsgeschichte. 1925.

DRUCKER, C., Editor.—Ostwald-Luther Hand- und Hilfsbuch zur Ausführung physikochemischer Messungen. Fourth edition revised. 1925.

ELLIS, CARLETON, and WELLS, ALFRED A.—Chemical Actions of Ultraviolet Rays. 1925.

EUCKEN, ARNOLD.—Fundamentals of Physical Chemistry. Translated and adapted from the second German edition by E. R. Jette and V. K. LaMer. 1925.

HAUSBRAND, E.—Principles and Practice of Distillation. Translated from the fourth new and enlarged German edition by E. H. Tripp. 1925.

HECK, ROBERT C. H.—Mechanics of Machinery. Part I: Mechanism. First edition. 1923.

Jahresbericht über die Leistungen der chemischen Technologie für das Jahr 1924. Jahrgang 70. Anorganischer Teil. 1925.

LANGTON, H. M.—Blacks and Pitches. 1925.

Machinery's Handbook for Machine Shop and Drafting-room. Sixth edition revised and enlarged. 1925.

MILLOT, STANISLAS.—Théorie Nouvelle de la Probabilité des Causes. 1925.

NEWMAN, F. H.—The Production and Measurement of Low Pressures. 1925.

Pharmacopœia of the United States of America. Tenth Decennial Revision. 1925.

PLANCK, MAX.—A Survey of Physics. No date.

Poor's Industrials. 1925. Volume two. 1925.

ROGERS, ALLEN, Editor.—Industrial Chemistry: A Manual for the Student and Manufacturer. Fourth edition. Two volumes. 1925.

SPROXTON, F.—Cellulose Ester Varnishes. 1925.

STAUDINGER, HERMANN.—Introduction to Qualitative Organic Analysis. 1925.

TAMMANN, GUSTAV.—The States of Aggregation. Translated from the second German edition by R. F. Mehl. 1925.

Thomas' Register of American Manufacturers. Sixteenth annual edition. 1925.

THOMSON, SIR J. J.—The Structure of Light. Fisan Memorial Lecture. 1925.

BOOK REVIEWS.

CHEMICAL SYNONYMS AND TRADE NAMES, by William Gardner, Works Chemist. Second edition, 271+56 pages, 8vo. London, Crosby Lockwood and Son; New York, D. Van Nostrand Company, 1925. Price, $8.

Notice was given of the first edition of this work and of a companion thereto, which was issued for the benefit of those who have the first edition,

a plan that is commendable, as was noted in the review of the supplement. In the present work we have a full second edition containing seventeen thousand definitions and cross-references, making a compact book of great value to workers in all departments of chemistry, pure and applied. The reviewer desires to repeat the suggestion given in the review of the companion volume, namely, that a loose-leaf system be adopted for certain types of works that need frequent revision. HENRY LEFFMANN.

CHEMICAL ENGINEERING CATALOG. Tenth annual edition, 1172 pages, illustrations, quarto. New York, The Chemical Catalog Company, Inc., 1925. Price, $10; leasing price, $2.

Notice was made in this JOURNAL for November, 1924, of the ninth edition of this comprehensive and valuable work. The present edition is on the same lines, with extensions and improvements, among which may be noted an enlarged classified directory in which for completeness the names of several hundred firms are given which do not take space in the catalog, an evidence of the open-mindedness and trustworthy character of the book. The section on technical and scientific books is also a most useful feature, enabling the user of the book to obtain easily a comprehensive view of the existing recent literature in many departments of chemistry. As noted in the review of the ninth edition, the work is a gratifying evidence of the organization and development that is prevailing in American chemical industries. HENRY LEFFMANN.

BLACKS AND PITCHES. By H. M. Langton, M.A., B.Sc. 179 pages, illustrated with diagrams and plates, 8vo. New York, D. Van Nostrand Company, 1925. Price, $4.50.

Though apparently covering but a limited field of applied chemistry, this book is quite a surprise in the revelation of the extent and intricacy of the processes for the manufacture and use of the black pigments. The principal forms are, as might be expected, more or less approaches to pure carbon. In addition to the presentation of the various forms of graphite, carbon-blacks and lampblack, much information is given in regard to asphalts and pitches obtained from the several forms of tar. In connection with graphite, note is made of the results obtained by Acheson which have enabled the production of artificial carbons essentially identical with the natural material. Outside of the blacks containing as a dominant ingredient carbon, black iron oxide and pyrolusite are mentioned, but, as the author says, they have no great importance. In connection with graphite several photomicrographs have been reprinted from C. Ainsworth Mitchell's book, showing the characters of the marks made by different kinds of pencil material. It is worth noting that our word "rubber," which is practically our only term for the elastic substances from plant latexes, originated in consequence of the use of such substances for erasing pencil marks.

Fixed carbon-blacks include the several forms of coal obtained by destructive distillation out of free air contact of animal and vegetable substances. These find many applications other than as pigments. Of late a specially active form of charcoal has been introduced, the "activated carbon," which has marked adsorptive powers. The bulk of the pigment black is of the soot

type, that is, collected from smoky flames. It is very striking to note the extent of this manufacture and the ingenuity that has been applied to it for securing the largest and best yields. Natural gas is one of the great sources.

Extended information is given concerning asphalts and petroleums. Many analyses of materials are tabulated. Considerable of the information is from American publications. This is especially true of the carbon-black industry, that is, the production from natural gas, the figures for 1920 (the latest given) showing an output of over fifty thousand pounds.

The book is one of a series of "Oil and Color Monographs."

HENRY LEFFMANN.

CELLULOSE ESTER VARNISHES. By F. Sproxton, B.Sc. xi–178 pages, 8vo. New York, D. Van Nostrand Company, 1925. Price, $4.50.

This work, like that noticed above, is part of the series of "Oil and Color Monographs." Though the subject to which it is devoted is a limited one, the data are extensive and very carefully set forth. A brief historical note gives an account of the development of the cellulose nitrate industry which has reached such great importance in both peace and war. Pelouze in 1838 investigated the action of nitric acid on cellulose products, such as linen and cotton, but Schönbein in 1845 prepared definite nitrates, keeping the process secret. Otto discovered and announced the method a little later. The fact that esters with noticeably different properties, especially as regards solubility, are obtainable with different strengths of acid and other modifications of manipulations was not at first clearly established, but now the several procedures are well understood. The use of these nitrates as high explosives is well known and was early introduced. In 1851, F. Scott Archer employed a solution in ether-alcohol as a vehicle for the coating of glass plates with sensitive silver halide emulsions, and fifty years ago the collodion (wet plate) method was in extensive use by photographers. Excellent results were obtained by it, but a disadvantage was that the plates must be used promptly. The introduction of the standard dry plate with the gelatin emulsion has caused the collodion process to be restricted to special lines.

The book in hand is not concerned with any other application of the cellulose esters, except that of solutions for varnishes. Two ester series are alone of commercial importance, the nitrates and the acetates. The latter have been slowly coming into use for motion picture films. It appears that the earliest development of the cellulose nitrate varnish occurred in the United States under the management of the Celluloid Company of New Jersey, which owned the patent granted in 1882 to J. H. Stevens (U. S. P. 269,340). Stevens discovered the applicability of amyl acetate as a solvent for these esters. Cellulose acetate was first obtained by Schützenberger in 1865.

The bulk of the text is, of course, devoted to the practical side and gives a large amount of valuable information concerning the varnishes. Theoretical data are not wholly ignored. A section is devoted to the molecular structure of cellulose. It has long been known that its percentage composition is identical with that of starch, but it has been fully evident that the molecule is much more complex than indicated by the simplest formula. This has been usually indicated by writing $(C_6H_{10}O_5)_n$. Some recent investigators have stated that

n must be at least 45. In addition to this discussion there is a considerable amount of information as to the nature of the varnishes, the data of physical chemistry being drawn upon for explanation. The book is therefore a comprehensive treatment of the field to which it is devoted.

HENRY LEFFMANN.

APROVECHAMIENTO DE LAS ENERGÍAS NATURALES. Estudio descriptivo y razonado de los modernos métodos que la técnica y la economía aconsejan para el aprovechamiento más perfecto posible de las fuentes naturales de energía. By Juan Gelpí Blanco, Engineer and Professor in the Barcelona Institute of Applied Physical Science. 283 pages, illustrations, 8vo. Barcelona, Libreria Subirana, 1924. Price, 24 pesetas.

The utilization of the energies of nature to the consideration of which this work is devoted is undoubtedly among the most important questions to which engineers can devote their investigations. The store of coal and oil is, indeed, still enormous, and there is no immediate danger of exhaustion, but as regards coal the labor question is becoming more and more serious and just at present one most important source of fuel is cut off without any encouraging indication of the resumption of the mining of it.

In his preface, the author calls attention to the interest that the nations are showing in developing their sources of power within their respective borders. The war, of course, stimulated such efforts. Barcelona, the city in which Professor Blanco teaches and works, is a centre of Spanish industry, and it is not surprising, therefore, that an extended and thorough study of natural sources of power should be undertaken by him. The book is both a practical and theoretical treatment of the topics. The fear that each great nation has concerning the danger of its dependence on others for any important supply has introduced, as the author points out, a new guiding principle in economics: "Bastarse a si mismo"—"Be self-sufficient."

Wind, waves, tides and waterfalls are among the topics considered. The fact that the work is written in Spanish must, of course, restrict somewhat its field of usefulness, but Spanish is really the best of the Continental tongues and would suit excellently as an international language. It is as nearly phonetic as a living language can be, its syntax is more simple than that of any other Continental tongue and there are few accented letters. The book in hand has no index of any real usefulness, but that lack is common among the book-makers of the Continent, although the Germans are now giving a great deal of attention to that feature. Professor Blanco's book is of much value and interest in the present state of economics. HENRY LEFFMANN.

INDUSTRIAL CHEMISTRY. A manual for student and manufacturer. By Allen Rogers, in conjunction with many collaborators. In two volumes, paged consecutively. Vol. 1, Inorganic. Vol. 2, Organic. Total pages, xliii–1267, with special index for Vol. 1, liberally illustrated, 8vo. New York, D. Van Nostrand Company, 1925. Price, $10.

It is noted on the title page of these volumes that the total issue has been twenty thousand, which is evidence of the merit of the work. This record has been made since 1912 when the first copyright was taken. A very large

list of collaborating specialists in the various industries is given, and works chemists, students and manufacturers may be assured that they have safe and sane guides in all the various applications of chemistry in large-scale operations. Liberal and good illustrations add to the serviceability of the work. The more important subjects are treated in considerable detail, for instance, the chapter on "Rubber" gives much information about the occurrence of the plants yielding latex. Similar thoroughness is seen almost everywhere in the work. It appears, however, that in connection with the glass industry there is a notable failure to speak of the methods of manufacture and the grades of the quartz glass now familiar in the market, which promise to be of such varied use and applications both in industries and in the laboratory. The article was written by a manufacturer of the standard material, and naturally he has no special interest or information about the subject of quartz glass. "Pyrex" does not appear in the index, and although an analysis of Jena glass is given, no data are furnished about the American product.

The paging of the two volumes, as noted, is consecutive. A useful arrangement is a separate index for Vol. 1, and then a complete index in Vol. 2. It seems to be time now to drop the medieval method of paging in Roman numerals and use the common numerals only, considering the title page as "1," and proceed from that to the end of the book.

The present edition will maintain the reputation and authority that the work has had since it was first issued.

HENRY LEFFMANN.

OSTWALD-LUTHER PHYSIKO-CHEMISCHE MESSUNGEN. Fourth edition, xx–814 pages, 510 illustrations, 8vo. Leipzig, The Akademische Verlagsgesellschaft m.b.H., 1925. Price, $8.35.

The first edition of this work appeared in 1893 under the authorship of W. Ostwald. It was in due time exhausted, but the occupation of the author in other departments of research delayed the issue of the second edition and R. Luther was joined in the work and has since carried on the further issues.

The book presents a most comprehensive and thorough summary of the methods of exact measurement in all departments of physical and chemical research. It is elaborately illustrated, mostly with very neat and clear line drawings which are more pleasing and more explanatory than the photogravures or wood-block illustrations now common in books on such topics. There is a tendency at the present day to illustrate text-books with cuts borrowed from catalogues of dealers. Considerable expense has no doubt been incurred in preparing the many drawings for the book, but it results in neatness of the page and explicitness of the data. An example of the thoroughness of the work is to be found in the section on stop-cocks. A great variety of these, adapted to all the purposes of the research worker, are figured. Liability of the plugs to become grooved thus causing leakage is pointed out. It is stated that quartz stops are now in the market. These will be a boon to the average laboratory worker, who has often to wrestle with a plug that has got stuck. The quartz stop can be heated without much risk.

The first edition of the book was issued in 1893, and was, therefore, what was then called "fin de siècle." Nordau satirized in his book "Entartung,"

the extended use of the phrase, and there was, indeed, much frivolous talk about the progress that had been made during the century and the basis of it. It is not part of the reviewer's duty to discuss the respective achievements of past times, but it is worthwhile to note that this book exemplifies one of the most striking features of the last century, namely, that it was specifically a period of development of methods of precision. Koch in his classic "Geschichte der Chemie," distinguished the fourth and final period of the history of chemistry as the "era of quantitative analysis," assigning its beginning about the time of the American Revolution, but the work of Lavoisier and Cavendish among others was the forerunner of much more extensive developments in accurate research. Berzelius by his researches into atomic weights contributed greatly to placing chemistry on an exact basis. The last half of the nineteenth century saw the perfection of the microscope and telescope, the establishment of the many units of energy, especially the placing of electrical phenomena on an exact basis, and the correlation of these with units of work. The work in hand came in the closing decade of the century, therefore most opportune, and its immediate success is not a matter of astonishment.

The present edition was delayed, as might be expected, by the war and has now appeared under the editorship of Carl Drucker, assisted by a corps of collaborators. In the general matter of book-making much praise must be given. The printing and paper are excellent; the excellence of the illustrations has already been noted. An extended table of contents and a comprehensive index add to the usefulness of the work. Compared with the prices now being asked for scientific books, the listed price seems quite reasonable.

HENRY LEFFMANN.

STATES OF AGGREGATION. The changes in the state of matter in their dependence upon pressure and temperature. By Gustav Tammann, University of Göttingen. Authorized translation by Robert Franklin Mehl, Ph.D. xi–295 pages, many illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $5.

The standard classification of the text-books on physics in vogue fifty years, by which matter was given three definite states, solid, liquid and gases, was never entirely satisfactory. *"Natura non facit saltum"* was a maxim of the older schools of scientists and philosophers. That "nature makes no leaps" is, indeed, evident in all fields of investigation. Transition conditions from solid to liquid and from liquid to gas are often observed. In the present work the gradation is carried further, for it is pointed out that the same substance can exist in several crystalline forms not reducible to one type and also as amorphous, that is, as a glass. The forms of carbon are often designated as allotropic, but it seems that this term should be limited to differences in the intimate molecular structure and not applied to the differences of arrangement of the molecules themselves as effective units in the mass. The most familiar instances of allotropism, oxygen and phosphorus, are dependent on atomic not on molecular groupings, but with graphite, diamond and carbon-blacks the differences are seemingly not regarded as due to changes in the intimate atomic structure.

The author of the book has been an especially industrious worker in the

study of states of aggregation. His first work "Crystallization and Fusion," met with good response, but the first edition of the present work under the title "Aggregationszustände" was much more successful. American work in the field has been very commendable, Professor Bridgman's researches having been notable for extending the observations to very high pressures. Bridgman presented some years ago before The Franklin Institute very striking results that had been obtained by the study of ice under such pressures.

In the investigations of this subject, the resources of chemistry have been drawn on freely. Organic chemistry has furnished many substances that have shown striking properties. The last section is devoted to the subject of liquid crystals, a somewhat paradoxical term. As might be expected, considerable difference of opinion has arisen on the subject. Liquids exist in three states. The first is the turbid mobile form, the second, termed "liquid crystals," because, upon cooling, their melts assume the character colors of turbid media. The third are the easily deformable weakly crystalline substances. Certain complex synthetic organic substances have been found especially applicable to investigations in this line.

It seems that this book should be of considerable value and interest to geologists and mineralogists. The phenomena of rock-formation and transformation are in many cases difficult to explain. As it is generally believed that the ingredients of the earth's crust have been subjected to enormous pressures and high temperatures, the effect of these upon the states of aggregation must have been considerable. The translation is well done.

HENRY LEFFMANN.

XIXe SALON INTERNATIONAL DE PHOTOGRAPHIE, Ce Salon Organisé par la Société Française de Photographie et la Photo Club de Paris, s'est tenu en l'Hôtel de la Société Française de Photographie, 51 rue de Clichy, Paris, du 3 au 14 Octobre, 1924. 48 photogravures, 4to. Paris, Society, 1924.

This volume illustrated with many very fine photogravures is the record of the exhibition of photographs held under the joint auspices of the Photo Club of Paris and the French Society of Photography. The carrying out of such exhibitions has been interrupted for several years, but the resuming has been quite encouraging, since artists from more than twenty nations have taken part. The system began in 1894; the interruption by the war was for ten years. The twentieth salon was held in October last. The subjects cover all fields: Portraiture, poses, landscape, village scenes, marines, incidents and architecture. There is, of course, considerable of the modern soft focus work which the scorners call "fuzzygraphs."

HENRY LEFFMANN.

ÉLÉMENTS DE LA TECHNIQUE DU PÉTROLE. By Georges Lévi. 100 pages, paper bound, illustrated, 8vo. Published by Les Matières Grasses, le Pétrole et ses Dérivés. Price, $1.25.

France, as the author aptly says, did not realize until a few years ago that it had a petroleum problem, but the war brought out the fact clearly. It has become, therefore, appropriate to present to the French people a simple state-

ment of the manner of the occurrence of petroleum and the methods of obtaining it. The book also contains information about the sanitary care of the workers and the methods of dealing with accidents that may occur at the wells and refineries. The author is a mining engineer connected with an institution at Strasburg, for instruction in the engineering questions concerning petroleum and other combustible liquids. The book is liberally illustrated with drawings, which, though mostly merely sketches, are satisfactory.

HENRY LEFFMANN.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS. Report No. 214, Wing Spar Stress Charts and Wing Truss Proportions. By Edward P. Warner. 18 pages, illustrations, quarto. Washington, Government Printing Office, 1925. Price, ten cents.

In order to simplify calculation of beams continuous over three supports, a series of charts has been calculated, giving the bending moments at all the critical points and the reactions at all supports for such members. Using these charts as a basis, calculations of equivalent bending moments, representing the total stresses acting in two bay-wing trusses of proportions varying over a wide range, have been determined, both with and without allowance for column effect. This leads finally to the determination of the best proportions for any particular truss or the best strut locations in any particular airplane. The ideal proportions are found to vary with the thickness of the wing section used, the aspect ratio, and the ratio of gap to chord.

Report No. 218, Standard Atmosphere—Tables and Data. By Walter S. Diehl. 28 pages, illustrations, quarto. Washington, Government Printing Office, 1925. Price, ten cents.

This report is an extension of Report No. 147. Detailed tables of pressures and densities are given for altitudes up to 20,000 metres and to 65,000 feet. In addition to the tables the various data pertaining to the standard atmosphere have been compiled in convenient form for ready reference.

PUBLICATIONS RECEIVED.

Industrial Chemistry. A manual for the student and manufacturer, edited by Allen Rogers. Fourth edition, two volumes, illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $10.

Standard Methods of Chemical Analysis, by Wilfred W. Scott. Fourth edition, two volumes, illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $12.

Cellulose Ester Varnishes, by F. Sproxton. 178 pages, illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $4.50.

Blacks and Pitches, by H. M. Langton. 179 pages, illustrations, plates, 8vo. New York, D. Van Nostrand Company, 1925. Price, $4.50.

Chemical Synonyms and Trade Names. A dictionary and commercial handbook by William Gardner. Second edition, 271 + 56 pages, 8vo. New York, D. Van Nostrand Company, 1925. Price, $8.

The States of Aggregation, by Gustav Tammann. Authorized translation from the second German edition by Robert Franklin Mehl. 279 pages, illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $5.

Photo-electricity—The Liberation of Electrons by Light, by H. Stanley Allen. Second edition, 320 pages, illustrations, 8vo. London, Longmans, Green and Company, 1925. Price, $6.50.

Zuckerchemie, by Hans Pringsheim. 322 pages, 8vo. Leipzig, Akademische Verlagsgesellschaft m.b.H., 1925. Price, 18 marks.

Hexosamines and Mucoproteins, by P. A. Levene. 163 pages, 8vo. London, Longmans, Green and Company, 1925. Price, $3.75.

Seventy-five Years of Gas Service in Chicago, by Wallace Rice. 58 pages, illustrated, 12 mo. Chicago, The Peoples Gas, Light and Coke Company, 1925.

National Advisory Committee for Aeronautics: Technical Notes, No. 228, The Flettner Rotor Ship in the Light of the Kutta-Joukowski Theory and of Experimental Results, by Frank Rizzo. 22 pages, plates, quarto. Washington, Committee, 1925.

Ontario Department of Mines: Thirty-second Annual Report, being Vol. 32, Part IV, 1923. Kirkland Lake Gold Area, by A. G. Burrows and Percy E. Hopkins. 96 pages, illustrations, maps, 8vo. Thirty-third Annual Report, being Vol. 33, Part V, 1924. Natural Gas in 1923 and Petroleum in 1923, by R. B. Harkness. 112 pages, illustrations, 8vo. Toronto, 1925.

The Department of the Interior: A review for 1925, by Hubert Work, Secretary of the Interior. 39 pages, illustrations, 8vo. Washington, Department of the Interior, 1925.

Éléments de la Technique du Pétrole (Recherche et Exploitation), par Georges Lévi, préface par M. Pineau. 100 pages, illustrations, 12mo. Paris, Editions de la revue "Les Matières Grasses," 1925. Price, 25 francs.

United States Department of Commerce: Scientific Papers of the Bureau of Standards, No. 511, A Non-intermittent Sensitometer (Time-scale Exposure Machine) with Clock-controlled Motor Drive, by Raymond Davis. 28 pages, illustrations, quarto. Scientific Papers, No. 512, Temperature Estimates of the Planet Mars, by W. W. Coblentz. 27 pages, diagrams, quarto. Technologic Papers, No. 291, Tests of Hollow Tile and Concrete Slabs Reinforced in One Direction, by D. E. Parsons and A. H. Stang. 52 pages, illustrations, plates, quarto. Technologic Papers, No. 293, Condensation of Water from Engine Exhaust for Airship Ballasting, by Robert F. Kohr. 42 pages, illustrations, plates, quarto. Technologic Papers, No. 296, Flow in a Low-carbon Steel at Various Temperatures, by H. J. French and W. A. Tucker. 24 pages, diagrams, plate, quarto. Washington, Bureau of Standards, 1925.

CURRENT TOPICS.

High Frequency Rays of Cosmic Origin. ROBERT ANDREWS MILLIKAN. (Copyright by Science Service, Inc.)—It was as early as 1903 that the British physicists, Rutherford and McLennan, noticed that the rate of leakage of an electric charge from an electroscope within an air-tight metal chamber could be reduced by enclosing the chamber within a completely encircling metal shield or box with walls a centimetre or more thick. This meant that the loss of charge must be due to highly penetrating rays, like the gamma-rays, which pass through metal walls and ionize the gas inside.

This radiation was provisionally called "penetrating radiation" and at first attributed to radio-active materials in the earth. In 1910 and 1911 it was found that it does not decrease with altitude as rapidly as it should on this view. A significant report was made by Gockel, a German physicist, who took an enclosed electroscope up in a balloon to 13,000 feet and found the "penetrating radiation" about as strong as at the surface, although Professor Eve, of McGill University, had calculated that it would fall to half its surface value at 250 feet.

In 1912, 1913, and 1914, Hess and Kohlhorster repeated the balloon measurements of Gockel, the latter going to 5.6 miles. They found the radiation decreasing a trifle for the first two miles and then increasing until at 5.6 miles it was eight times as great as at the surface.

This indicated that the penetrating rays come from out the earth. The war put a stop to further studies, but as soon as proper instruments could be built in the newly equipped Norman Bridge Laboratory of Physics, Millikan and Bowen sent up at Kelly Field in 1922, near San Antonio, Texas, four little recording electroscopes in sounding balloons to almost twice the height formerly attained. The highest point was near ten miles. These instruments were built to hold 300 c.c. of air at 150 pounds pressure, and provided with recording barometer, thermometer, and electroscope, three different sets of moving photographic films and the necessary driving mechanism. The total weight was 180 grams.

Millikan and Bowen expected to find very large rates of discharge; nine-tenths of the atmosphere was beneath, and only one-tenth was left to cut down the intensity of the rays from outside. The results were contrary, but it was proved that the penetrating radiation was greater at great altitudes than at the surface, but that the amount of the increase was not more than a fourth of that expected. In 1924 Hess and Kohlhorster reduced their estimates to accord with the above measurements.

The origin of the rays was uncertain, with indications of a cosmic source. Millikan and Otis studied how penetrating are the rays.

Since they are weaker at the surface, they went to the top of Pike's Peak in the summer of 1923, carrying up 300 pounds of lead and a $6 \times 6 \times 6$ foot water-tank for the sake of making absorption measurements on such rays as were found at that altitude.

They found that the electroscopes discharged twice as fast on Pike's Peak as at Pasadena, but the rays were cut down so fast by the absorbing screens that the greater part were not appreciably more penetrating than the gamma-rays; also that the discharge of the electroscope was decreased 10 per cent. by a heavy snow-storm. This showed that the chief part of the rays are of local origin, and might be due to radio-active matter in the upper atmosphere.

The cause of the increase with altitude of these rays became quite as interesting as the question of a penetrating radiation of cosmic origin, since this latter would produce at the most but a small fraction of the observed increase between Pasadena and Pike's Peak. Millikan and Cameron planned experiments for 1925 which were designed (1) to settle the question of the existence of small, penetrating radiation of cosmic origin, and (2) to throw light on the cause of the variation with altitude of the softer radiation of the gamma type which had been found more than twice as intense on Pike's Peak as at Pasadena. To do this it was necessary to find at high altitudes deep snow-fed lakes, for any radio-active contamination of the water through its seepage through the earth would vitiate the results obtained by sinking electroscopes to different depths beneath the surface of the lake.

The first experiment was made at Muir Lake (11,800 feet) hundreds of feet deep, under the brow of Mount Whitney. Here for the last ten days in August, they sank electroscopes at various depths to sixty feet, and brought to light altogether unambiguously a cosmic radiation of such penetrating power that the electroscope reading kept decreasing to a depth of forty-five feet. The atmosphere over the lake was equivalent in absorbing power to twenty-three feet of water, indicating rays, coming into the earth from outer space, that could pass through 45 plus 23 equalling 68 feet of water or the equivalent of six feet of lead, before being completely absorbed. This represents rays much harder (more penetrating) than any which had before been imagined.

The most penetrating X-rays produced in hospitals do not go through half an inch of lead. The rays originating out in space are at least a hundred times more penetrating. High penetrating power means, according to modern physics, high frequency or short wave-length. The experiments indicate a region of frequencies as far above X-rays as these are above the frequencies of ordinary light waves.

These high-frequency rays are not homogeneous, but have a measurable spectral distribution, the shortest observed being a little less than twice the frequency of the longest, for the rays which were actually observed in Muir Lake changed hardness or frequency as

they were filtered through greater and greater thicknesses of water, just as X-rays are successively hardened by passing through successive layers of lead. The experiments with the sounding balloons indicate that the frequencies of these cosmic rays do not extend over into the X-ray region, else larger discharges would have occurred in the experiments with sounding balloons, when nine-tenths of the atmosphere had been left beneath. Good evidence was obtained that these rays shoot through space in all directions, as no change was noted in their intensity day or night.

The results obtained in Muir Lake were checked with wonderful completeness by observations in another snow-fed lake—Arrowhead—300 miles from Muir, 7000 feet lower, and equally deep. The absorbing power of the atmosphere between the elevations of Muir and Arrowhead Lakes is the equivalent of about two metres of water, and as a matter of fact every reading in Arrowhead was practically identical with one taken in Muir at a depth two metres lower.

The most penetrating rays that have been known, the gamma-rays of radium and thorium, are produced by nuclear transformations within atoms. They are produced by the change of one atom over into another, or by the creation of a new type of atom. These still more penetrating rays are probably also produced by nuclear transformations of some sort, but such transformations must be enormously more energetic than those in any known radio-active changes. The frequency of any emitted ray is, according to present knowledge, proportional to the energy of the sub-atomic change which produces it. Nuclear changes having an energy value perhaps fifty times as great as the energy changes involved in observed radio-active processes are probably taking place through space, and signals of these changes are being sent to us in high-frequency rays.

The energy of the nuclear change which corresponds to the formation of helium out of hydrogen is known, and from it can be computed the corresponding frequency. It is found to correspond closely to the highest frequency rays observed this summer. The computed frequencies of these rays also correspond closely to the energy involved in the simple capture of an electron by a positive nucleus. This phenomenon may be going on through space and be the most probable source of the rays. It is true that the formula underlying the computation of frequencies from their absorption coefficient is uncertain. It is a formula, nevertheless, that works well in the frequency range in which independent checks are possible, namely, in the X-ray and gamma-ray fields. According to this formula the wave-length of the shortest waves here investigated is 0.0004 Ångström, or but one-fiftieth of that of the hardest gamma-rays known, and but one ten-millionth that of ordinary light. The longest wave-length found is about five-thirds of the shortest, or 0.00067 Ångström.

When these rays strike the earth they should be partially transformed, according to the well-established Compton effect, into soft

scattered rays about the hardness, or wave-length, of the rays observed at Pike's Peak and Mount Whitney. The reason these were more plentiful on the mountain peaks than at Pasadena, is that there are more than twice as many of the hard rays to be transformed at the peaks than at Pasadena. This seems to be the solution of the second summer's problems.

How can nuclear transformation, for example, as formation of helium out of hydrogen or the capture of an electron by a positive nucleus, be going on all through space, the resulting rays coming apparently as much from one direction as from any other, and certainly not more plentifully from the direction of the sun than from a diametrically opposite point, as shown by the equality of midday and midnight data? The difficulty is not insuperable, in view of the transparency of large amounts of matter for these hard rays, combined with Hubble's recent proof at the Mount Wilson Observatory that some of the spiral nebulæ are at least a million light years away. The centres at which these nuclear changes are taking place would then only have to occur at extraordinarily widely scattered intervals to produce the intensity of the radiation observed at Muir Lake.

The only alternative hypothesis might seem to be that the observed rays are generated in the upper layers of the atmosphere by electrons shooting through space in all directions with practically the speed of light. This hypothesis might help in interpreting the fact of the maintenance of the earth's negative charge, but it meets with insuperable obstacles, in explaining quantitatively the variation with altitude of the ionization in closed vessels. In any case, this hypothesis is very much like the one represented above, it fills space with rays travelling in all directions with the speed of light. From some such conception there seems to be no escape, and yet it is one which is almost too powerful a stimulus to the imagination. Professor MacMillan will see in it evidence of the condensation of light and heat into matter out somewhere in space. The psychists will be explaining all kinds of telepathic phenomena by it.

The experiments seem to point to the following conclusions: (1) That these extraordinarily penetrating rays exist; (2) that their mass absorption coefficient may be as high as 0.18 per metre of water; (3) that they are not homogeneous, but are distributed through a spectral region far up above X-ray frequencies—probably 1000 times the mean frequencies of X-rays; (4) that upon striking matter these hard rays stimulate softer rays of about the frequency predicted by the theory of the Compton effect; (5) that these rays come into the earth with equal intensity day and night and at all hours of the day or night, and with practically the same intensity in all directions. H. L.

The Magnetization of the Lavas from Mount Etna and the Direction of the Magnetic Field in Sicily in the Twelfth and Thirteenth Centuries. R. CHEVALLIER. (*Ann. de Phys.*, July-

Aug., 1925.)—The influence of masses of rock upon the compass seems to have been noticed first by Joao de Castro in the case of the island of Chaul off Bombay. Humboldt in 1797 made a detailed study of the magnetic state of Heidelberg Mountain. Half a century later A. Delesse showed that a large number of eruptive rocks can be permanently magnetized and that such magnets show the sequence of poles when broken. It was, however, still believed somewhat commonly that rocks containing iron compounds were only temporarily magnetized. This view was supported by the fact that it was difficult to get a magnetized rock to repel a compass needle. Melloni took up this problem in 1853 and, using his astatic magnetoscope, showed conclusively that 108 kinds of rock from different European countries were permanently magnetized. He attributed to the influence of the earth's magnetic field the magnetism that he had detected in the rocks. Just as in the United States the upper end of each picket in an iron fence is a south magnetic pole owing to the earth's influence, so he found that the upper faces of recent lava flows on Mount Vesuvius were south poles. He further heated fragments of lava to a red heat and let them cool slowly as lava cools in volcanic regions. The cooled specimens he found to be magnetized by the terrestrial field. During the last decade of the nineteenth century Folgheraiter pursued the same line of investigation. As a result of his examination of the magnetic state of Greek and Etruscan vases he concluded that the magnetic inclination, small in amount and negative in sign, at the eighth century B.C., grew in angle and attained a value of 60° by the first century of our present era. From his work emerged the fact of the extraordinary permanence of the magnetism acquired by ceramic material at high temperatures. From this property joined to the necessity of having the vertical axes of such articles as vases and jugs occupy a definite position in the kiln, it became possible to trace the changes of inclination through the centuries. Since, however, these vessels might be turned about their vertical axes into any position while being fired, no conclusion concerning the declination of the earth's field can be derived from their magnetic state conserved throughout the ages. This lack in regard to the declination would disappear provided that the lavas retained as obstinately as the clays the magnetism which they had upon cooling, for the sheets of lava have not been greatly shifted with the passing of time. Brunhes about the beginning of the present century took up this question and found that the clays baked by sheets of lava in Auvergne, France, and the sheets themselves, had the same directions of magnetization. Since the permanence of the magnetism of the clays was established by Folgheraiter, the permanence of that of the lavas followed no matter whether there happened to be a neighboring sheet of clay as a guarantee or not. In one case Brunhes found the polarity of the upper surface of a sheet of basalt to be north, thus presenting confirmation of Folgheraiter's conclusion that at one time the inclination in Europe was nega-

tive, that is, that then the south pole of the dipping needle pointed downward.

The author of the present article sets himself the task of examining the magnetic data obtainable from the lavas of Mount Etna, which have not been hitherto studied. He is limited in his work by an absence of definite dates for any flows that came between those of 256 and 1284 B.C. His circumscribed problem is therefore to find the direction of the terrestrial field in Sicily from the twelfth century onwards. The lavas from seven or eight flows were investigated, though the identification of one from the time of the occupation of the island by the Carthaginians was not certain.

The curve connecting the dates of eruptions with the inclination of the corresponding lavas does not at all agree with what is known of the secular course of this quantity in Europe. This irregularity is attributed to the effect of other lava sheets lying below those under discussion. For the declination the results are more regular. "These lava flows within historic time give a curve of declination that is the exact prolongation of the curve known from direct measurement. It passes, going from the present to the past, from western to eastern declination about 1630 A.D., reaches an eastern maximum of 18° near 1440 and again becomes western near 1240, thus displaying a period of 750 years, if the phenomenon really continues to be periodic. The eastern and western branches of the curve without having the same form do have the same amplitude of 18° and the same length of 375 years." It is interesting to note that in London the declination changed from the earlier eastern to the present western direction in 1660 A.D.

A new way of dating a lava flow is suggested. "The flow of Sona, which various arguments locate between the decline of Rome and the Norman invasion, corresponds from the magnetic point of view exactly to the eruption of 812 and not to that of 1169 as has been supposed."

A boundless field of the prehistory of terrestrial magnetism is unfolding itself. So far as the reviewer remembers, no work of this kind has been done for the North American continent. G. F. S.

The Scattering of Light from Liquid Boundaries. C. V. RAMAN and L. A. RAMDAS. (*Proc. Roy. Soc.*, A. 748.)—The surfaces of liquids have attained fresh interest now that Hardy and Langmuir have shown them to consist, in the case of organic liquids at least, of fine orderly molecules, all standing at attention. The authors are desirous of learning whether this molecular layer is doubly refracting or not, how the arrangement of the surface molecules is affected by thermal agitation and whether "the surface layer is capable of producing (by reason of its thermal agitation or its optical anisotropy or other cause) the observed elliptic polarization of the light reflected by liquid surfaces (clean or contaminated as the case may be) at the Brewsterian angle." The present paper is but

the first of a series and is mainly concerned with scattering from mercury.

To get a clean surface of the liquid, mercury already cleaned, washed and dried was introduced into a bulb joined by a tube to a second bulb. After exhaustion the apparatus was sealed. By successive distillations whatever dust was still present was collected in one bulb while the surface of the mercury in the other was uncontaminated. By an achromatic lens, sunlight was focussed on the surface. The observations had to do, not with the regularly reflected light, but with that which was diffused from the illuminated surface.

Some interesting observations were made. When unpolarized light fell normally on the surface, "the light scattered in a direction grazing the liquid surface is almost completely polarized, the principal component of the electric vector being then perpendicular to the surface. The notable feature is thus that the light scattered through 90° has its electric vector perpendicular to the wave-front of the incident light, instead of being parallel to it as in the Rayleigh scattering by small particles." When unpolarized light just grazed the liquid surface, the light scattered backward toward the point from which the light came was much stronger than that scattered forward in the direction of the reflected beam.

The light scattered from mercury surfaces produces the effect of a bluish-white opalescence. Its intensity is small. "A clean mercury surface illuminated normally with unpolarized light in the green part of the solar spectrum and viewed at 45° is found to have only $5.7/10^{7}$ times the brightness of a plaster-of-Paris surface viewed under the same conditions." A surface of liquid sodium, examined in the Norman Bridge Laboratory, presented the same effects as mercury.

G. F. S.

A Study of the Electrolytic Interrupter. J. A. CROWTHER and R. J. STEPHENSON. (*Phil. Mag.*, July, 1925.)—In the Wehnelt interrupter investigated a wire protruded through the wall of a glass tube into an electrolyte, through which the electric current passed to a plate of lead, or of some other indifferent material. A solution of H_2SO_4 is commonly used, though ammonium phosphate often replaces it, because no noxious fumes are evolved. The interruptions of the current are both sharp and regular. The wire in contact with the electrolyte bears a bluish glow. A revolving mirror showed this to be intermittent, and further examination with a falling photographic plate showed the light to correspond to the breaking of the circuit.

In the interrupter a flow of current begins when the voltage applied between the point and the lead plate exceeds the e.m.f. of polarization. As the applied voltage increases, the current increases linearly until the voltage across the terminals is 12.5 volts for a platinum point in dilute sulphuric acid. A further increase of the voltage was followed by a decrease of the current, the measuring

ammeter and voltmeter meanwhile fluctuating greatly. This variable stage continued until the applied voltage reached about 35. For this the current was approximately one-ninth as much as at 12.5 volts. A steady state has now been reached, and for further increments of voltage the current grows less linearly. Through this last stage the point was usually free from any glow. The critical voltage, 12.5, remained the same when the length of the wire projecting into the liquid or its diameter was doubled, though the current for the critical voltage increases in proportion to the area of the wire in contact with the electrolyte. It is therefore the nature of the electrolyte that fixes the critical voltage. Similar relations of voltage to current were discovered for sodium hydrate, sodium chloride, sodium sulphate, hydriodic acid, and ammonium phosphate. For the last named the critical voltage was 45. When iron, nickel or lead was substituted for platinum, the critical voltage seemed to be a little less. These metals disintegrate so promptly that they cannot replace platinum in practical work. During the first stage, when voltage and current are increasing together, gas was given off from the wire in bubbles of moderate size. In the third stage, where voltage and current changed in opposite directions, the gas clung closely to the wire, working its way to the upper part and breaking loose in large bubbles. In the intermediate stage bubbles characteristic of the first and of the third stages were developed, but, in addition, many tiny bubbles were shot away from the wire with considerable speed. These rendered the liquid milky in appearance for some time. "It is natural to postulate the formation of a highly resistant layer at the anode, and the tightly clinging gas layer which is so characteristic of the high-resistance (the third) stage of the conduction presents itself as a very obvious explanation." What takes place in a Wehnelt interrupter with inductance in the circuit is as follows: "When the circuit is made, the voltage across the break gradually rises, until, after a time depending on the time constant of the circuit, it reaches a value exceeding the critical value. The current then begins to fall rapidly, and an induced e.m.f. is produced which rapidly attains the sparking voltage, while the current falls to a small value. A discharge then takes place across the resistant layer, thus destroying it and bringing the system back to the normal stage. The current then begins to rise again, and the cycle is repeated indefinitely. The regularity of the interruptions is also explained, since the time which it takes for the voltage across the break to rise to a definite value will be determined by the applied voltage and the time constant for the circuit." The authors know well that much still awaits explanation in the operation of the interrupter, but they have made a valuable contribution toward a full understanding of the process. G. F. S.

Potassium as a Mercury-vapor Trap. A. L. HUGHES and F. E. POINDEXTER. (*Phil. Mag.*, Aug., 1925.)—High vacua can be reached quickly and simply by the use of the mercury-vapor condensation

pump. There is, however, the disadvantage that the vapor of mercury is not removed as are the other gases and vapors. To keep the mercury vapor from finding its way into the vessel to be exhausted it has been customary to insert a trap surrounded by liquid air between the pump and the vessel. If this trap for weeks at a time had to receive a supply of liquid air, it became a burden of care and of expense. The present paper is the record of a successful quest for something to replace liquid air as a barrier to mercury vapor. "For practical purposes, it is fair to say that the diffusion of mercury vapor past a certain region in a vacuum system is stopped by lining a trap with an alkali metal quite as satisfactorily as by cooling the trap with liquid air."

In one experiment a discharge tube for spectroscopic tests was joined to a reservoir containing 52 grams of mercury through a vessel on the inner walls of which had been deposited 2.5 grams of potassium. The area covered by the metal was 30 sq. cm. The discharge tube contained helium. Not until the eleventh week did any mercury lines appear along with the helium lines. By that time 3.7 grams of mercury had evaporated. "A potassium-lined trap will act as a barrier to mercury vapor until the potassium has absorbed at least 150 per cent. of its own weight of mercury."

A comparison was made between the pressure attained with a liquid air-trap and that reached with a potassium-lined trap. In the former case the pressure was 143×10^{-9} mm., and in the latter 74×10^{-9} mm. Even after contact with air had turned the metallic lustre of the potassium surface into a dull black, the metal still possessed its power of absorbing mercury vapor. Indeed it was with potassium thus darkened that the lowest pressure was reached with a certain gauge, *viz.*, 10×10^{-9} mm.

Any alkali metal will serve to absorb the mercury vapor, but sodium and potassium are the cheapest. The authors give a technique for manipulating these metals so as to avoid introducing into the apparatus any constituent of the oil in which these metals come commercially. In one instance an entire system, McLeod gauge, ionization gauge, pump and trap had to be discarded because of the unwelcome presence of something contained in the protecting oil.

G. F. S.

The Effect of Infra-red Radiation upon the Rate of Combustion of Inflammable Gaseous Mixtures. W. T. DAVID. (*Proc. Roy. Soc.*, A 748.)—Two plans were used to provide the radiation whose effect was to be determined. In the first radiation from an electrically heated coil entered the explosion chamber through either a quartz or a fluorite window. In the second the inner surface of the chamber was silver plated, so that the radiation resulting from the explosion surged back and forth from the reflecting surface. The progress of the explosion was followed by means of pressure-time records. In the first method comparisons were made of records fur-

nished by mixtures of identical composition, receiving in one case, and in the other lacking, radiation entering through the window. In the second method comparisons were made between records obtained with the inner surface silvered and, on the other hand, with it blackened and consequently absorbing. The inflammable mixtures used were hydrogen and air, carbon monoxide and air, and methane and air.

With the hydrogen mixture neither plan of furnishing radiation made any considerable difference. In the case of CO more rapid combustion manifested itself when the inner surface was polished and, again, when radiation entered through the fluorite. When the window was of quartz what radiation passed through it into the chamber made no difference in the rapidity of combustion. Finally, with the methane more rapid combustion was found when either quartz or fluorite covered the window, but silvering the inner surface made no difference except to cause a slight increase of pressure near the end of combustion. A critical examination of these results in the light of a knowledge of the wave-lengths absorbed by the several mixtures led to this conclusion: "An increased rate of combustion as indicated by the pressure-time curves was found in all cases when the superimposed radiation (first type) or increased radiation (second type) was of a kind which could be absorbed by the combustible gases." For example, methane does not absorb the radiation emitted by its combustion products and, hence, prolonging by silvering the time during which the methane was exposed to radiation that it could not absorb could scarcely affect the results. The reason why a quartz window in the case of CO renders ineffective the radiation from the coil is that the mineral absorbs those radiations that the gas can absorb.

The increase of pressure due to the absorption of radiation sometimes amounts to as much as 30 per cent. G. F. S.

Forest Fires in France.—Americans are so familiar with reports of forest fires in their own country and hear so much about scientific forest management in Europe, that it is worth noting an article in *Revue Scientifique* concerning such fires in France. Noted forests, such as Fontainebleau, suffer frequently. Much is said, much is written, very long official reports are published, but it is doubtful if the problem is dealt with practically. F. Michotte, an engineer, has given close attention to the problem for some years. The following are the statistics for a year: January, none; February, 1; March, 2; April, 5; May, 1; June, 3; July, 13; August, 55; September, 35; October, 1, November and December, none. These figures suggest that the principal cause of such fires is the same in France as in the United States, namely, the carelessness of campers, for the greater number occur in the outing season. It must be borne in mind that in Western Europe the incidence of the vacation season is a little later than in America. Almost all the fires are recorded as occurring in the vacation months. Of course, then the trees are in leaf and fire

can spread easily. Michotte offers suggestions for fire-prevention, some of which are usual, such as clearing out brush, supervision by wardens and rangers and watch-towers. He regards the area assigned to a watcher in the United States as too great. As France, like the rest of Western Europe, is much more densely populated than most of the large forest areas of the United States, it appears that in the former country forest fires are public spectacles, the announcement of a fire drawing large numbers of automobile tourists.

Concerning the causes of fires, Michotte thinks that smokers and carelessness do not play a great part; spontaneous combustion and the action of lightning are important causes. He does not seem to give any notice to the work of campers, who are usually considered as frequent starters of forest fires in America. The spontaneous combustion of hay-mows was asserted by Michotte in a paper offered to the Academy of Sciences in 1912, which was refused, because the view was then unpopular. This point has especial application at the present time when numerous barn fires have occurred in the districts around Philadelphia, which experts have declared are due to such spontaneous combustion, usually of alfalfa, but which the general public mostly ascribe to incendiaries. It is, however, to be borne in mind that at this time, illicit stills are probably frequently the cause.

H. L.

Production of Gold in 1924.—The world's gold mining last year was not widely active, the Transvaal alone showing increased yield, being in fact one of its best years. The output was 9,597,643 ounces fine, valued at about $180,000,000. The total production of the year was less than double the African yield. About 71 per cent. was from British Empire territory. The United States produced about two million and a half ounces, California, Colorado, Alaska, South Dakota, Arizona and Nevada being the contributing states, the relative amounts being the order given. Canada produced about one million and a half ounces, principally from Ontario. France itself produced no appreciable amount, but small yields were obtained in Algiers and Madagascar. It is most unfortunate that the platinum production does not reach such figures. It is interesting to note that the British Empire has a strong hold on certain natural productions, such as gold and rubber, but the economic problem of the greatest importance to British statesmen is to secure supplies of oil. It appears from the above figures that British territory yielded gold enough to purchase the oil necessary for the operation of British ships and other oil-using industries, but while conditions of peace permit such easy exchange, war would seriously alter these opportunities and gold cannot be a substitute for any other raw material of importance.

H. L.

Blue Coal.—Mechanical engineers, especially those that give particular attention to combustion problems, are fully aware of the term "white coal" that Continental engineers have introduced as meaning

water-power. The application of tidal movement as a source of power has been a hope for a long while, especially since the development of electric transmission. The utilization of the waves has also been a matter of study. *Revue Scientifique* states that a tide-motor is in successful operation in Finistére, and that the wave-motor is still more promising. It briefly notes an experimental apparatus that has been installed at Monaco. A trumpet-shaped pipe receives the wave and directs it against the blades of a turbine operating a dynamo. A number of such pipes arranged in parallel can be installed. In very stormy weather it is necessary to close the pipes by doors. It is suggested that wind-motors might be installed along the coasts, using a method of concentrating the air movement on the blades of turbines, similar to those used in the wave-motors. To this use of wave motion the French journal applies the term "blue coal." H. L.

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INDEX TO VOL. 200.

Acoustic impedance, The measurement of (Kennelly), 467

Allotropy of zinc, An investigation of the alleged (Peirce, Anderson and Van Dyck), 349

Alpha-particles, Theory of the range of (Loeb and Condon), 595

Ammonia. Impact experiments in compound gases (Waldie), 507

ANDERSON, E. A., W. M. PEIRCE and P. VAN DYCK: An investigation of the alleged allotropy of zinc by X-ray analysis and a redetermination of the zinc lattice, 349

Atmospheric electric observations taken at the University of Chicago during the total eclipse of January 24, 1925 (Coade and Merrymon), 497

Atmospheric observations taken at the University of Minnesota during the time of the eclipse of January 24, 1925 (Zeleny), 501

Atmospheric observations taken at Yale University during the total eclipse of January 24, 1925 (Swann), 489

Aviation engine performance (Sparrow), 711

Biochemical study of by-product yeast (Hepburn), 767

BOOK REVIEWS:

Anderson, R. J.: The metallurgy of aluminium and aluminium alloys, 694

Armsby, H. P., and C. R. Moulton: The animal as a converter of matter and energy, 410

Arrhenius, S. A.: Chemistry in modern life, 696

Bardorf, C. F.: The story of sugar, 556

Bardorf, C. F., and J. A. B. Ball: The elements of sugar refining, 557

Bazant, Z.: Influence du système de triangulation sur les efforts secondaires, 266

Beal, Sarah: Astronomic determination by the U. S. Coast and Geodetic Survey, 558

Blanco, J. G.: Aprovechamiento de las energías naturales, 829

Bloch, E.: The kinetic theory of gases, 702

Bragg, W. H.: The crystalline state, 403

Bureau of Standards: Radio instruments and measurements, 269

Campbell, C. A. R.: Bats, mosquitoes and dollars, 697

Central High School of Philadelphia, Handbook, 2nd edition, 411

Chemical engineering catalog, ed. 10, 1925, 827

Christiansen, W. G.: Organic derivatives of antimony, 268

Crenshaw, B. H., and C. D. Killebrew: Analytic geometry and calculus, 267

XIXe Salon de photographie, 822

Ellis, C., and A. A. Wells: The chemical action of ultra-violet light, 700

Eucken, A.: The fundamentals of physical chemistry, 699

Faraday Society: Base exchange in soils, 401

Faraday Society: The physical chemistry of igneous rock-formation, 560

Foster, W.: The elements of chemistry, 690

Frehafer, M. K., and C. L. Snow: Tables and graphs for facilitating the computation of spectral-energy distribution by Planck's formula, 703
Friend, J. N.: A text-book of inorganic chemistry, vol. 3, part 1, The alkaline earth metals, 406
Galbrun, H.: Assurances sur la vie, calcul des primes, 143
Gardner, William: Chemical synonyms and trade names, 826
Gardner, William: Companion to the first edition of chemical synonyms and trade names, 403
Guye, C. E.: Physico-chemical evolution, 142
Hodkin, E. W., and A. Cousen: A text-book of glass technology, 693
Judge, Arthur W.: The testing of high-speed internal combustion engines, 405
Langton, H. M.: Blacks and pitches, 827
Lecat, M.: Bibliographie de la Relativité, 402
Lévi, G.: Éléments de la technique du pétrole, 832
Lodge, Oliver: Electrical precipitation, 559
Marx, E.: Handbuch der radiologie, vi, 553
Matagrin, A.: L'Industrie des produits chimiques et ses travailleurs, 409
Millot, S.: Théorie nouvelle de la probabilité des causes, 691
Montel, P.: Statique et résistance des matériaux, 141
National Advisory Committee for Aeronautics, Reports: No. 200, 143; No. 208, 270; No. 210, 411; No. 212, 271; No. 214, 833; No. 215, 702; No. 216, 412; No. 218, 833
Newell, L. C.: College chemistry, 139
Nielsen, N.: Traité élémentaire des nombres de Bernoulli, 404
Olivier, C. P.: Meteors, 142
Ostwald-Luther physiko-chemische Messungen, ed. 4, 830
Pearson, W. A., and J. S. Hepburn: Physiological and clinical chemistry, 692
Physics in industry, vol. 2, 404
Planck, M.: A survey of physics, 701
Reynaud-Bonin, E.: Appareils et installations telephoniques, 551
Rogers, A.: Industrial chemistry, ed. 4, 829
Service, J. H., and G. E. Frease: A laboratory manual of machine shop practice, 270
Sproxton, F.: Cellulose ester varnishes, 828
Staudinger, H.: Introduction to qualitative organic analysis, 703
Tammann, G.: States of aggregation, 831
Taylor, H. S.: A treatise on physical chemistry, 407
Towles' portrait lighting, 552
Zipprich, A. E.: Freehand drafting, 140
Zsigmondy, R.: Kolloidchemie, ed. 5, 555
BRIDGMAN, P. W.: Certain aspects of high-pressure research, 147
Cathode-ray oscillograph, A note on the (Terroux), 771
CHASE, CLEMENT E.: Research and experimental tests in connection with the design of the bridge over the Delaware River between Philadelphia and Camden, 417
CHU, C. T.: Soft X-rays from certain metals, 615
COADE, E. N., and WM. M. MERRYMON: Atmospheric electric observations taken at the University of Chicago during the total eclipse of January 24, 1925, 497

COBB, PERCY W., and FRANK K. MOSS: Eye fatigue and its relation to light and work, 239
COBLENTZ, W. W., and C. O. LAMPLAND: Some measurements of the spectral components of planetary radiation and planetary temperatures, 103
Color production, A theory of (Stieglitz), 35
CONDON, EDWARD, and LEONARD LOEB: Theory of the range of alpha-particles, 595
CURRENT TOPICS: 33, 49, 86, 102, 126, 128, 132, 135, 136, 182, 217, 234, 238, 247, 253, 259, 264, 272, 273, 311, 326, 333, 361, 362, 390, 391, 395, 400, 413, 414, 488, 496, 499, 503, 518, 538, 542, 547, 594, 607, 614, 627, 633, 672, 676, 680, 684, 705, 707, 738, 746, 756, 765, 766, 774, 814, 815, 816, 820, 835
DAY, ARTHUR L.: Some causes of volcanic activity, 161
Delaware River Bridge between Philadelphia and Camden, Research and experimental tests in connection with the design of the (Chase), 417
Delaware River Bridge between Philadelphia and Camden, The towers, cables and stiffening trusses of the (Moisseiff), 436
Drying of wood, A mathematical theory of the (Tuttle), 609
Eastman Kodak Company Research Laboratory Notes: 127, 391, 535, 673, 815
Electrons in gases, Motion of (Townsend), 563
Engineering, Some future problems in (Parsons), 219
Engineering, Military (Jadwin), 1
ERIKSON, HENRY A.: Observations on residual ionization in a closed vessel taken at Biwabic, Minnesota, during the solar eclipse of January 24, 1925, 505
EVE, A. S.: On recent advances in wireless propagation both in theory and in practice, 327
Eye fatigue and its relation to light and work (Cobb and Moss), 239
FRANKLIN INSTITUTE:
Bartol Research Foundation, Notes: 823
Centenary addresses:
Bridgman, P. W.: Certain aspects of high-pressure research, 147
Day, Arthur L.: Some causes of volcanic activity, 161
Humphreys, W. J.: The way of the wind, 279
Jacobus, D. S.: Stimulation of research and invention, 249
Jadwin, Edgar: Modern military engineering, 1
Kennelly, A. E.: The measurement of acoustic impedance with the aid of the telephone receiver, 467
Stieglitz, Julius: A theory of color production, 35
Townsend, J. S.: Motion of electrons in gases, 563
Franklin medal presentation, 363
Library notes, 139, 265, 401, 550, 689, 826
Membership notes, 138, 265, 401, 549, 687, 824
Science and the Arts Committee, Abstract of meeting: June 3, 1925, 137; October 7, 1925, 687; November 4, 1925, 821
Stated meeting, proceedings: October 21, 1925, 685; November 18, 1925, 821
FRY, THORNTON C.: The theory of the schroteffekt (correction), 819
Gases, compound, Impact experiments in (Waldie), 507
Gases, Motion of electrons in (Townsend), 563
Heaviside's operational mathematics (Smith), 519, 635, 775
HEPBURN, JOSEPH S.: A biochemical study of by-product yeast, 767

HERING, CARL: Magnetic flux energy, 747
H-functions (Smith), 519, 635, 775
HENDERSON, E. P., and FRANK L. HESS: Polycrase from Brazil, 235
HESS, FRANK L., and E. P. HENDERSON: Polycrase from Brazil, 235
High-pressure research, Certain aspects of (Bridgman), 147
HUMPHREYS, W. J.: The way of the wind, 279
Hydrogen peroxide, The action of, on single-layer silver halide plates (Wightman, Trivelli and Sheppard), 335
Impact experiments in compound gases (Waldie), 507
INGRAM, W. H.: The theory of the telephone receiver, 757
Invention, Stimulation of research and (Jacobus), 249
Ionization, residual, Observations on (Erikson), 505
JACOBUS, D. S.: Stimulation of research and invention, 249
JADWIN, EDGAR: Modern military engineering, 1
KENNELLY, A. E.: The measurement of acoustic impedance with the aid of the telephone receiver, 467
LAMPLAND, C. O., and W. W. COBLENTZ: Some measurements of the spectral components of planetary radiation and planetary temperatures, 103
Latent image, Formation of the (Sheppard, Trivelli and Loveland), 51
LOEB, LEONARD, and EDWARD CONDON: Theory of the range of alpha-particles, 595
LOVELAND, R. P., S. E. SHEPPARD and A. P. H. TRIVELLI: Studies in photographic sensitivity, vi, 51
LUCKIESH, M., and FRANK K. MOSS: The rate of visual work on alternating fields of different brightnesses, 731
Magnetic flux energy (Hering), 747
Magnetization of spectrum lines (Zeeman), 305
Mathematical theory of the drying of wood (Tuttle), 609
Mathematics, Operational, of Heaviside (Smith), 519, 635, 775
MERRYMON, WM. M., and E. N. COADE: Atmospheric electric observations taken at the University of Chicago during the total eclipse of January 24, 1925, 497
Metals, Soft X-rays from certain (Chu), 615
Military engineering, Modern (Jadwin), 1
MOISSEIFF, LEON S.: The towers, cables and stiffening trusses of the bridge over the Delaware River between Philadelphia and Camden, 436
MOSS, FRANK K., and PERCY W. COBB: Eye fatigue and its relation to light and work, 239
MOSS, FRANK K., and M. LUCKIESH: The rate of visual work on alternating fields of different brightnesses, 731
Paint vehicles and pigments, Spectro-photometric measurements of (Stutz), 87
PARSONS, WILLIAM BARCLAY: Some future problems in engineering, 219
PEIRCE, W. M., E. A. ANDERSON and P. VAN DYCK: An investigation of the alleged allotropy of zinc by X-ray analysis and a redetermination of the zinc lattice, 349
Photographic sensitivity, Studies in, vi (Sheppard, Trivelli and Loveland), 51
Photographic sensitivity, Studies in, vii (Wightman, Trivelli and Sheppard), 335
Planetary radiation and temperatures, Some measurements of the spectral components of (Coblentz and Lampland), 103

Polycrase from Brazil (Hess and Henderson), 235
Publications received, 145, 271, 412, 561, 704, 833
Radiation, Solar, during the total eclipse of January 24, 1925 (Seemann), 629
Range of alpha-particles, Theory of the (Loeb and Condon), 595
Relativity, The principle of (Swann), 183
Research and invention, Stimulation of (Jacobus), 249
Saha's equation for temperature ionization, A derivation of (Swann), 591
Schroteffekt, The theory of the (Fry), 819
SEEMANN, HERMAN E.: Solar radiation during the total eclipse of January 24, 1925, 629
SHEPPARD, S. E., A. P. H. TRIVELLI and R. P. LOVELAND: Studies in photographic sensitivity, vi, The formation of the latent image, 51
SHEPPARD, S. E., E. P. WIGHTMAN and A. P. H. TRIVELLI, Studies in photographic sensitivity, vii, 335
Silica, fused, The mechanical, thermal and optical properties of (Thomson), 313
Silver halides, X-ray analysis of some mixed crystals of the (Wilsey), 739
SMITH, J. J.: An analogy between pure mathematics and the operational mathematics of Heaviside by means of the theory of H-functions, 519, 635, 775
Solar eclipse, January 24, 1925, Atmospheric electric observations taken at the University of Chicago during the (Coade and Merrymon), 497
Solar eclipse, January 24, 1925, Atmospheric electric observations taken at Yale University during the (Swann), 489
Solar eclipse of January 24, 1925, Observations on residual ionization in a closed vessel, taken at Biwabic, Minnesota, during the (Erikson), 505
Solar eclipse, January 24, 1925, Observations on the variations of the potential gradient of the atmosphere taken at the University of Minnesota during the time of the (Zeleny), 501
Solar eclipse, January 24, 1925, Solar radiation during the (Seemann), 629
Solar radiation during the total eclipse of January 24, 1925 (Seemann), 629
SPARROW, STANWOOD W.: Aviation engine performance, 711
Spectral components of planetary radiation and planetary temperatures (Coblentz and Lampland), 103
Spectro-photometric measurements of paint vehicles and pigments in the ultra-violet (Stutz), 87
Spectrum lines, Magnetization of (Zeeman), 305
STIEGLITZ, JULIUS: A theory of color production, 35
STUTZ, GEORGE F. A.: Observations of spectro-photometric measurements of paint vehicles and pigments in the ultra-violet, 87
SWANN, W. F. G.: Atmospheric electric observations taken at Yale University during the total eclipse of January 24, 1925, 489
SWANN, W. F. G.: A derivation of Saha's equation for temperature ionization, 591
SWANN, W. F. G.: The principle of relativity, 183
Telephone receiver, The theory of the (Ingram), 757
Temperature ionization, A derivation of Saha's equation for (Swann), 591
TERROUX, F. RICHARD: A note on the cathode-ray oscillograph, 771

THOMSON, ELIHU: The mechanical, thermal and optical properties of fused silica, 313
TOWNSEND, J. S.: Motion of electrons in gases, 563
TRIVELLI, A. P. H., S. E. SHEPPARD and R. P. LOVELAND: Studies in photographic sensitivity, vi, 51
TRIVELLI, A. P. H., E. P. WIGHTMAN and S. E. SHEPPARD: Studies in photographic sensitivity, vii, 335
TUTTLE, FORDYCE: A mathematical theory of the drying of wood, 609
U. S. Bureau of Chemistry Notes, 129, 255, 393, 539, 677, 817
U. S. Bureau of Mines Notes, 133, 261, 397, 543, 681
U. S. Bureau of Standards Notes, 377
VAN DYCK, P., E. P. WIGHTMAN and A. P. H. TRIVELLI: An investigation of the alleged allotropy of zinc by X-ray analysis and a redetermination of the zinc lattice, 349
Visual work, Rate of, on alternating fields of different brightnesses (Luckiesh and Moss), 731
Volcanic activity, Some causes of (Day), 161
WALDIE, ALMA T.: Impact experiments in compound gases. Ammonia, 507
WIGHTMAN, E. P., A. P. H. TRIVELLI and S. E. SHEPPARD: Studies in photographic sensitivity, vii. The action of hydrogen peroxide on single-layer silver halide plates, 335
WILSEY, R. B.: X-ray analysis of some mixed crystals of the silver halides, 739
Wind, The way of the (Humphreys), 279
Wireless propagation, On recent advances in (Eve), 327
Wood, A mathematical theory of the drying of (Tuttle), 609
X-ray analysis of some mixed crystals of the silver halides (Wilsey), 739
X-ray analysis of zinc (Peirce, Anderson and Van Dyck), 349
X-rays, Soft, from certain metals (Chu), 615
Yeast, A biochemical study of by-product (Hepburn), 767
ZEEMAN, PIETER: Magnetization of spectrum lines. Reminiscences and prospects, 305
ZELENY, ANTHONY: Observations on the variations of the potential gradient of the atmosphere taken at the University of Minnesota during the time of the eclipse of January 24, 1925, 501
Zinc, An investigation of the alleged allotropy of (Peirce, Anderson and Van Dyck), 349